Entropy and the Shattering Dimension

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The Shattering dimension of a class is a real-valued version of the Vapnik-Chervonenkis dimension. We will present a solution to Talagrand’s entropy problem, showing that the $L_2$-covering numbers of every uniformly bounded class of functions are exponential in the shattering dimension of the class. Formally we prove that there are absolute constants $K$ and $c$ such that for every $0 < t \leq 1$ and any probability measure $\mu$, $N(t, F, L_2(\mu)) \leq (2/t)^{K\text{vc}(F,ct)}$, where $F$ is a class of functions bounded by 1 and $\text{vc}(F, t)$ is the shattering dimension at scale $t$. This extends a result of Dudley from the Boolean case to the real-valued one.

This result has many applications in the theory of empirical processes (e.g. identifying classes which satisfy the uniform CLT using their shattering dimension), convex geometry (optimal Elton’s Theorem on sign-embeddings of $\ell_1^n$, and the fact that if the Euclidean entropy of a convex body is ”large” at a scale $t$ then there exists a high-dimensional cube of side $ct$ contained in a coordinate projection of the body) and in Learning Theory (improved generalization bounds in terms of the shattering dimension).