Proc. R. Soc. Lond. B **226**, 303–323 (1985)
Printed in Great Britain

Ill-posed problems in early vision: from computational theory to analogue networks

By T. Poggio and C. Koch
Massachusetts Institute of Technology, Artificial Intelligence Laboratory and Center for Biological Information Processing, 545 *Technology Square, Cambridge, Massachusetts* 02139, *U.S.A.*

(*Communicated by S. Brenner, F.R.S. – Received* 5 *September* 1984 – *Revised* 13 *May* 1985)

We outline a theoretical framework that leads from the computational nature of early vision to algorithms for solving them and finally to a specific class of analogue and parallel hardware for the efficient solution of these algorithms. The common computational structure of many early vision problems is that they are mathematically ill-posed in the sense of Hadamard. Regularization analysis can be used to solve them in terms of variational principles of a specific type that enforce constraints derived from a physical analysis of the problem. Studies of human perception may reveal whether principles of a similar type are exploited by biological vision. We also show that the corresponding variational principles can be implemented in a natural way by analogue networks. Specific electrical and chemical networks for localizing edges and computing visual motion are derived. We suggest that local circuits of neurons may exploit this unconventional model of computation.

1. Introduction

One of the best definitions of early vision is that it is inverse optics: a set of computational problems that both machines and biological organisms have to solve. While in classical optics the problem is to determine the images of physical objects, vision is confronted with the inverse problem of recovering three-dimensional shape from the light distribution in the image. Most processes of early vision such as stereo matching, computation of motion and all the 'structure from' processes can be regarded as solutions to inverse problems. This common characteristic of early vision can be formalized: most early vision problems are 'ill-posed problems' in the sense of Hadamard. In this article we will show that a mathematical theory developed for regularizing ill-posed problems leads in a natural way to the solution of early vision problems in terms of variational principles of a certain class. This is a theoretical framework for some of the variational solutions already obtained in the analysis of early vision processes. It shows how several other problems in early vision can be approached and solved. In addition, it suggests a new model for visual computation that may be biologically significant. A large gap exists at present between computational theories of vision and their possible implementation in neural hardware. The model of computation provided by the digital computer is clearly unsatisfactory for the neurobiologist, given the increasing evidence that neurons are complex devices, very different from the simple digital switches as portrayed by the McCulloch & Pitts (1943) type of threshold neurons. It is especially difficult to imagine how networks of neurons may solve the equations involved in vision algorithms in a way similar to digital computers. We suggest an analogue model of computation in electrical or chemical networks for a large class of vision problems, that maps more easily into biologically plausible mechanisms. Ill-posed problems in early vision can be 'solved', according to regularization theories, by variational principles of a specific type. A natural class of system for solving these variational problems are electrical, chemical or neuronal networks. We show how to derive specific networks for solving several low-level vision problems, such as the computation of visual motion and edge detection.

2. Variational solutions to vision problems

In recent years, the computational approach to vision has begun to shed some light on several specific problems. One of the recurring themes of this theoretical analysis is the identification of physical constraints that make a given computational problem determined and solvable. Some of the early and most successful examples are the analyses of stereo matching (Marr & Poggio 1976, 1979; Grimson 1981*a*, *b*; Mayhew & Frisby 1981; Nishihara 1984; Kass 1984; for a review, see Nishihara & Poggio 1984) and structure from motion (Ullman 1979*a*, *b*). In these studies constraints such as continuity of three-dimensional surfaces in the case of stereo matching and rigidity of objects in the case of structure from motion play a critical role for obtaining a solution.

More recently, variational principles have been used to introduce specific physical constraints. A variational principle defines the solution to a problem as the function that minimizes an appropriate cost function. Many problems can be formulated in this way, including laws that are normally expressed in terms of differential equations. In physics, for instance, most of the basic laws have a compact formulation in terms of variational principles, that require the minimization of a suitable functional, such as the Lagrangian for classical mechanics.

For instance, the problem of interpolating visual surfaces through sparse depth data can be solved by minimizing functionals that embed a constraint of smoothness (Grimson 1981*b*, 1982; Terzopoulos 1983). Thus, the surface that best interpolates the data minimizes a certain cost functional which measures how much the surface deviates from smoothness. Computation of the motion field in the image can be successfully performed by finding the smoothest velocity field consistent with the data (Horn & Schunck 1981; Hildreth 1984*a*, *b*): in other words among all possible velocity fields that are consistent with the data a solution can be found by choosing the velocity field that varies the least. In a similar way, shape can be recovered from shading information in terms of a similar variational method (Ikeuchi & Horn 1981). The computation of subjective contours (Ullman 1976; Brady *et al.* 1980; Horn 1981), of lightness (Horn 1974) and of shape from contours (Barrow & Tennenbaum 1981; Brady & Yuille 1984) can also be formulated in terms of variational principles. Terzopoulos (1984, 1985) has recently reviewed the use of a certain class of variational principles in vision problems within a rigorous theoretical framework.

We wish to show that these variational principles follow in a natural and rigorous way from the ill-posed nature of early vision problems. We will then propose a general framework for 'solving' many of the processes of early vision.

3. Ill-posed problems

Hadamard first defined a mathematical problem to be well-posed when its solution (i) exists; (ii) is unique and (iii) depends continuously on the initial data (notice that for the solution to be robust against noise in practice the problem must be not only well-posed but also well-conditioned).

Most of the problems of classical physics are well-posed, and Hadamard argued that physical problems had to be well-posed. 'Inverse' problems, however, are usually ill-posed. Inverse problems can be obtained from the direct problem by exchanging the role of solution and data. Consider, for instance,

$$y = Az, \tag{1}$$

where A is a known operator. The direct problem is to determine y from z, the inverse problem is to obtain z when y ('the data') is given. Though the direct problem is usually well-posed, the inverse problem is usually ill-posed, when z and y belong to a Hilbert space.

Typical ill-posed problems are analytical continuation, back-solving the heat equation, super-resolution, computer tomography, image restoration and the determination of the shape of a drum from its frequency of vibration, a problem which was made famous by Marc Kac (1966). In early vision, most problems are ill-posed because the solution is not unique (but see later the case of edge detection), since the operator corresponding to A is usually not injective, as in the case of shape from shading, surface interpolation and computation of motion (see Poggio & Torre 1984; Bertero *et al.* 1986).

4. Standard regularization methods

Most ill-posed problems are not sufficiently constrained. To regularize them and make them well-posed, one has to introduce generic constraints on the problem. In this way, one attempts to force the solution to lie in a subspace of the solution space, where it is well defined. The basic idea of regularization methods is to restrict the space of acceptable solutions by choosing the function that minimizes an appropriate functional. Specific and rigorous regularization theories for 'solving' ill-posed problems have been developed during the past years (see especially Tikhonov 1963; Tikhonov & Arsenin 1977; Nashed 1974, 1976). We will refer to all techniques, that involve the minimization of a quadratic functional, as standard regularization theory. The regularization of the ill-posed problem of finding z from the data y such that $Az = y$ requires the choice of norms $\|\cdot\|$ (usually quadratic) and of a stabilizing functional $\|Pz\|$. The choice is dictated by mathematical considerations, and, most importantly, by a physical analysis of the generic constraints on the problem. Standard regularization theory consists of three main methods (Poggio & Torre 1984).

(i) Among z that satisfy $\|Pz\| \leqslant C$, where C is a constant, find z that minimizes

$$\|Az-y\|. \tag{2}$$

(ii) Among z that satisfy $\|Az-y\| \leqslant \epsilon$, find z that minimizes

$$\|Pz\|. \tag{3}$$

(iii) Find z that minimizes

$$\|Az-y\|^2+\lambda\|Pz\|^2, \tag{4}$$

where λ is a regularization parameter ($\lambda = \epsilon/C$ where ϵ and C are used in equations (2) and (1), respectively).

The first method consists of finding the function z that satisfies the constraint $\|Pz\| \leqslant C$ and best approximates the data. The second method computes the function z that is sufficiently close to the data (ϵ depends on the estimated errors and is zero if the data are noiseless) and is most 'regular'. In the third method, the regularization parameter λ controls the compromise between the degree of regularization of the solution and its closeness to the data. Regularization theory provides techniques to determine the best λ (Tikhonov & Arsenin 1977; Wahba 1980). It also provides a large body of results about the form of the stabilizing functional P that ensure uniqueness of the result and convergence. For instance, it is possible to ensure uniqueness in the case of Tikhonov's stabilizing functionals (also called stabilizers of pth order) defined by

$$\|Pz\|^2 = \int \sum_{r=0}^{p} c_r(\xi)\left(\frac{\mathrm{d}^r z}{\mathrm{d}\xi^r}\right)^2 \mathrm{d}\xi, \tag{5}$$

where $c_r(\xi)$ are non-negative weighting factors. Equation (5) can be extended in natural way to several dimensions. If one seeks regularized solutions of (1) with P given by (5) in the Sobolev space W_2^p of functions that have square-integrable derivatives up to pth order, the solution can be shown to be unique and stable under mild conditions (the data must be given at a set of points that defines a unique functional in the null space of the smoothness functional), if A is linear and continuous. It turns out that most stabilizing functionals used so far in early vision are of the Tikhonov type (see also Terzopoulos 1984). They all correspond to either interpolating or approximating splines (for method 2 and method 3, respectively).

5. Example 1: motion

The claim by Poggio & Torre (1984) is that variational principles introduced recently in early vision for the problem of shape from shading, computation of motion, and surface interpolation are exactly equivalent to the standard regularization techniques we described. The associated uniqueness results are directly provided by regularization theory. We briefly discuss the case of motion computation in its recent formulation by Hildreth (1984a, b).

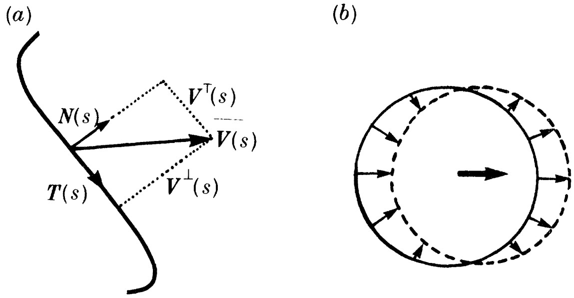

FIGURE 1. Decomposition and ambiguity of the velocity field. (*a*) The local velocity vector $\boldsymbol{V}(s)$ in the image plane is decomposed according to (6) into components perpendicular and tangent to the curve. (*b*) Local measurements cannot measure the full velocity field: the circle undergoes pure translation: the arrows represent the perpendicular components of velocity that can be measured from the images. Redrawn from Hildreth (1984*a*).

Consider the problem of determining the two-dimensional velocity field along a contour in the image. Local motion measurements along contours provide only the component of velocity in the direction perpendicular to the contour. The component of velocity tangential to a smooth contour is invisible to a local detector that examines a restricted region of the contour. Figure 1 shows how the local velocity vector $\boldsymbol{V}(s)$ is decomposed into a perpendicular and a tangential component to the curve

$$\boldsymbol{V}(s) = v^{\top}(s)\,\boldsymbol{T}(s) + v^{\perp}(s)\,\boldsymbol{N}(s). \tag{6}$$

The component $v^{\perp}(s)$ and direction vectors $\boldsymbol{T}(s)$ and $\boldsymbol{N}(s)$, are given directly by the initial measurements, the 'data'. The component $v^{\top}(s)$ is not and must be recovered to compute the full two-dimensional velocity field $\boldsymbol{V}(s)$. Thus the 'inverse' problem of recovering $\boldsymbol{V}(s)$ from the data $v^{\perp}$ is ill-posed because the solution is not unique. Mathematically, this arises because the operator K defined by

$$\boldsymbol{V}\cdot\boldsymbol{N} = K\boldsymbol{V} \tag{7}$$

is not injective. Equation (7) describes the imaging process as applied to the physical velocity field $\boldsymbol{V}$ which consists of the x and y components of the velocity field on the image plane.

Intuitively, the set of measurements given by $v^{\perp}(s)$ over an extended contour should provide considerable constraint on the motion of the contour. An additional generic constraint, however, is needed to determine this motion uniquely. For instance, rigid motion on the plane is sufficient to determine $\boldsymbol{V}$ uniquely but is very restrictive, since it does not cover the case of motion of a rigid object in space. Hildreth (1984*a*) suggested, following Horn & Schunck (1981), that a more general constraint is to find the smoothest velocity field among the set of possible velocity fields consistent with the measurements. The choice of the specific form of this constraint was guided by physical considerations (the real world consists of solid objects with smooth surfaces whose projected velocity field is usually smooth) and by mathematical considerations, especially uniqueness of the solution. Hildreth proposed two algorithms: in the case of exact data the functional to be minimized is a measure of the smoothness of the velocity field

$$\|P\boldsymbol{V}\|^2 = \int \left(\frac{\partial \boldsymbol{V}}{\partial s}\right)^2 \mathrm{d}s \tag{8}$$

subject to the measurements $v^{\perp}(s)$. Since in general there will be error in the measurements of $v^{\perp}$, the alternative method is to find $\boldsymbol{V}$ that minimizes

$$\|K\boldsymbol{V} - v^{\perp}\|^2 + \lambda \int \left(\frac{\partial \boldsymbol{V}}{\partial s}\right)^2 \mathrm{d}s. \tag{9}$$

It is immediately seen that these schemes correspond to the second and third regularizing method, respectively (the first regularizing method corresponds, with the same P, to rigid translation in the image plane). Uniqueness of the solutions, proved by Hildreth for the case of (8), is a direct consequence for both (8) and (9) of standard theorems of regularization theories. In addition, other results can be used to characterize the behaviour of the correct solution depending on the smoothing parameter λ.

6. Example 2: edge detection

Poggio *et al.* (1985) have recently applied standard regularization techniques to another classical problem of early vision: edge detection. Edge detection, intended as the process that attempts to detect and localize changes of intensity in the image (this definition does not encompass all the meanings of edge detection) is a problem of numerical differentiation (Torre & Poggio 1985). Notice that differentiation is a common operation in early vision and is not restricted to edge detection. The problem is ill-posed because the solution does not depend continuously on the data. The problem is to find the solution z to $y = Az$ with $(Az)(x) = \int_0^x z(s)\,\mathrm{d}s$. Thus, z is the derivative of the data y.

The intuitive reason for the ill-posed nature of the problem can be seen by considering a function $f(x)$ perturbed by a very small (in L_2 norm) 'noise' term $\epsilon \sin \Omega x$. The terms $f(x)$ and $f(x) + \epsilon \sin \Omega x$ can be arbitrarily close for very small ϵ, but their derivatives may be very different if Ω is large enough. This simply means that a derivative operation 'amplifies' high-frequency noise.

In one dimension, numerical differentiation can be regularized in the following way. The 'image' model is $y_i = f(x_i) + \epsilon_i$, where y_i is the data and ϵ_i represent errors in the measurements. We want to estimate f'. We choose a regularizing functional $\|Pf\| = \int (f''(x))^2\,\mathrm{d}x$, where f'' is the second derivative of f. This choice corresponds to a constraint of smoothness on the intensity profile. Its physical justification is that the (noiseless) image is smooth because of the imaging process: the image is a band-limited function and has therefore bounded derivatives. The second regularizing method (no noise in the data) is equivalent to using interpolating cubic splines for differentiation. The third regularizing method, which is more natural since it takes into account errors in the measurements, leads to the variational problem of minimizing (see Rheinsch 1967)

$$\Sigma(y_i - f(x_i))^2 + \lambda \int (f''(x))^2 \, \mathrm{d}x. \tag{10}$$

Poggio *et al.* (1985) have shown (i) that the solution $f(x)$ of this problem can be obtained by convolving the data y_i (assumed on a regular grid and satisfying appropriate boundary conditions) with a convolution filter R, and (ii) that the filter R is a cubic spline with a shape very close to a gaussian and a size controlled by the regularization parameter λ (see figure 2). Differentiation can then be accomplished by convolution of the data with the appropriate derivative of this filter. The optimal value of λ can be determined for instance by cross-validation and other techniques. This corresponds to finding the optimal scale of the filter (see Poggio & Torre 1984; Poggio *et al.* 1985).

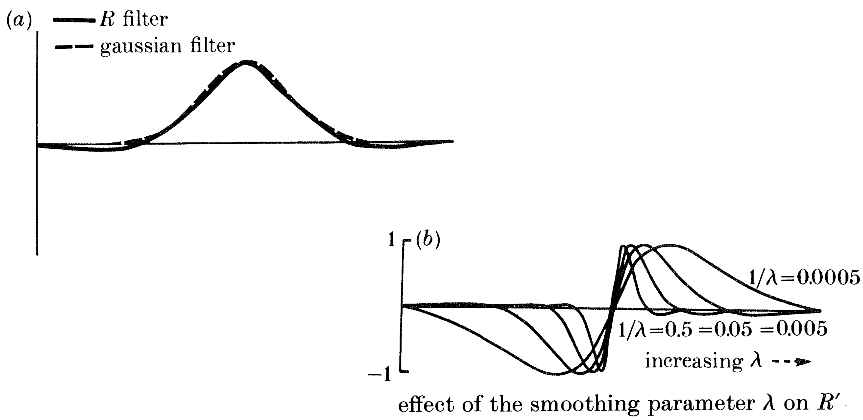

FIGURE 2. The regularized edge detection filter. (*a*) The convolution filter obtained by regularizing the ill-posed problem of edge detection with method (III) (solid line; Poggio *et al.* 1985). It is a cubic spline, very similar to a gaussian (dotted line). (*b*) The first derivative of the filter for different values of the regularizing parameter λ, which effectively controls the scale of the filter. This one-dimensional profile can be used for two-dimensional edge detection by filtering the image with oriented filters with this transversal cross-section and choosing the orientation with maximum response (see Canny 1983). The second derivative of the filter (not shown here) is quite similar to the second derivative of a gaussian.

These results can be directly extended to two dimensions to cover both edge detection and surface interpolation and approximation. The resulting filters are very similar to the derivatives of a gaussian extensively used in recent years (Marr & Poggio 1979; Marr & Hildreth 1980; see also Canny 1983 and Torre & Poggio 1985).

Other problems in early vision such as shape from shading (Ikeuchi & Horn 1981) and surface interpolation (Grimson 1981*b*, 1982; Terzopoulos 1983; Blake 1984*b*), in addition to the computation of velocity, have been formulated and 'solved' in similar ways using variational principles of the type suggested by regularization techniques (although this was not realized at the time). Other problems such as stereo and structure from motion can be approached in terms of regularization analysis (see Poggio & Torre 1984).

7. Physical plausibility of the solution

Uniqueness of the solution of the regularized problem, which is ensured by formulations such as equations (2)–(4), is not the only (or even the most relevant) concern of regularization analysis. Physical plausibility of the solution is the most important criterion. The decision regarding the choice of the appropriate stabilizing functional cannot be made judiciously from purely mathematical considerations. A physical analysis of the problem and of its generic constraints play the main role. Standard regularization theory provides a framework within which one has to seek constraints that are rooted in the physics of the visual world. This is, of course, the challenge of regularization analysis.

In our example of the computation of motion the constraint of smoothness is justified by the observation that the projection of three-dimensional objects in motion onto the image plane tends, in a probabilistic sense, to yield smooth velocity fields (see Hildreth 1984*a*). In the case of edge detection the constraint of a small norm for the derivative of image intensity is directly motivated by the band-limiting properties of the optics. In the case of motion, however, and more dramatically in the case of surface reconstruction, the constraint of smoothness is not always correct. This suggests that more general stabilizing functionals are needed to deal with the general problem of discontinuities (see Conclusion).

A method for checking the physical plausibility of a variational principle is, of course, computer simulation. A simple general technique we suggest here is to use the Euler–Lagrange equation associated with the variational problem. In the computation of motion, Yuille (1983) has obtained the following sufficient and necessary condition for the solution of the variational principle (8), to be the correct physical solution

$$\boldsymbol{T}\cdot(\partial^2\boldsymbol{V}/\partial s^2) = 0$$

where $\boldsymbol{T}$ is the tangent vector to the contour and $\boldsymbol{V}$ is the true velocity field. The equation is satisfied by uniform translation or expansion and by rotation only if the contour is polygonal. These results suggest that algorithms based on the smoothness principle will give correct results, and hence be useful for computer vision systems, when (i) motion can be approximated locally by pure translation, rotation or expansion, or (ii) objects have images consisting of connected straight lines. In other situations, the smoothness principle will not yield the correct velocity field, but may yield one that is qualitatively similar and close to human perception (Hildreth 1984*a*, *b*). In the corresponding case for edge detection (intended as numerical differentiation), the solution is correct if and only if the intensity profile is a polynomial spline of appropriate degree.

From a more biological point of view, a careful comparison of the various 'regularization' solutions with human perception promises to be a very interesting area of research, as suggested by Hildreth's work on the computation of motion. For some classes of motions and contours, the solution of (8) and (9) is not the physically correct velocity field. In these cases, however, the human visual system also appears to derive a similar, incorrect velocity field (Hildreth 1984*a*, *b*).

8. Analogue networks for solving variational problems

As suggested by Terzopoulous (1984*a*), analogue networks, chemical, electrical or mechanical, are a natural computational model for solving variational principles. We know from physics that the behaviour of such systems, in fact the behaviour of any physical system, can be described by using variational principles (MacFarlane 1970). In the frictionless world of classical mechanics a system's state variables will behave in such a way as to minimize the associated Lagrangian. Electrical network representations have been constructed for practically all of the field equations of physics, many of them are equivalent to variational principles (for an electrical network implementation of Schrödinger's equation see Kron (1945)). A fundamental reason for the natural mapping between variational principles and electrical or chemical networks is Hamilton's least action principle (for more details see appendix 1).

The class of variational principles that can be computed by analogue networks is dictated by Kirchhoff's current and voltage laws, which simply represent conservation and continuity restrictions satisfied by each network component (appropriate variables are usually voltage and current for electrical networks and affinity, that is, chemical potential, and chemical turnover rate for chemical systems). Kirchhoff's current and voltage laws provide the unifying structure of network theory. A large body of theoretical results is available about networks satisfying them, including classical thermodynamics (Oster *et al.* 1971). Notice that Kirchhoff's laws are physical restatements of the topological properties of the dynamic space. For electrical networks they correspond to conservation of flows (Kirchhoff's current law) and uniqueness of potential (Kirchhoff's voltage law).

For a network containing only sources and linear resistances, Hamilton's least action principle implies Maxwell's minimum heat theorem: the distribution of voltages and currents is such that it minimizes the total power dissipated as heat. These results can be extended to nonlinear circuit components (MacFarlane 1970; Oster & Desoer 1971; Poggio & Koch 1984) but in the following we will restrict ourselves to linear resistances (possibly negative, see appendix 3). Consider for simplicity a network of discrete elements. The power dissipated by each linear resistance in the circuit is a quadratic term of the form

$$I_k V_k, \tag{11}$$

where I_k is the current through and V_k the voltage difference across the resistive process r_k. It follows that any network consisting of linear resistances and voltage sources E_i minimizes the following associated quadratic functional

$$\sum_k r_k I_k^2 - \sum_i E_i I_i, \tag{12a}$$

where the second sum includes all the batteries. For a network of resistances and current sources I_i the functional that the network minimizes is given by

$$\sum_k g_k V_k^2 - \sum_i I_i V_i, \tag{12b}$$

where the second sum includes all the current sources and $g_k = 1/r_k$. In the limit, as the meshes of the circuit become infinitesimally small, the network solves the continuous variational problem, and not simply its discrete approximation.

It is then easy to show the equivalence of (12) and the standard regularization principle equation (4). A proof is given in appendix 2. Thus, electrical networks of linear resistances and batteries (or current sources) can solve quadratic variational principles of the form of (4). The solution is unique when (4) yields a unique solution (which is usually the case, see Poggio & Torre (1984)).

Electrical networks of resistances and batteries do not have any dynamics. In practice, however, small capacitances will always be present and the stability of the network must then be considered. It turns out that networks implementing regularization principles of the form of (4) are indeed stable, under the same conditions that ensure a unique solution (see appendix 3).

An equivalent way to see how electrical networks can implement variational principles of the form of (4) is to consider the associated Euler–Lagrange equations. Since the functional to be minimized is quadratic, the Euler–Lagrange equations are linear, of the form $Qz = b$ (see appendix 2 for a definition of Q). They have a unique solution z, corresponding to the unique solution of the variational principle. In the discrete case, these equations correspond to n linear, coupled algebraic equations. We claim that these equations can be implemented in a network containing only linear resistances and sources, where the vector b, which depends on the data, can always be represented in terms of current or voltage sources. The matrix Q corresponds to the symmetrical, real matrix of the network resistances. To see this, one sets up one mesh for every variable z_i (with the associated mesh current I_i). Each mesh consists of a battery E_i in series with one resistance r_i and $n-1$ resistances of some constant value k. Moreover, a simple auxiliary circuit connects the ith and the jth mesh via an auxiliary resistance $R_{ij} = R_{ji}$ for every non-zero entry q_{ij} in Q. The associated circuit current flows not only through R_{ij}, but also through the resistance k of the ith mesh and k of the jth mesh. The values of the resistances are then given by $r_i = \Sigma_l q_{il} - (n-1)k$ and $R_{ij} = R_{ji} = -k(k/q_{ij}+2)$. The scheme requires exactly as many resistances r_i and R_{ij} as there are non-zero entries in Q. Although this procedure will always yield an electrical network with linear elements implementing $Qz = b$, its physical realization might require negative resistances.

As pointed out in the context of vision (earlier, Horn (1974) proposed an analogue implementation of the lightness computation and Ullman (1979*b*) a relaxation technique working in locally connected simple networks for the shape from motion problem) a significant advantage of analogue networks is their extreme parallelism and speed of convergence (Koch *et al.* 1985). Furthermore, resistive networks are robust against random errors in the values of the resistances (Karplus 1958). A disadvantage is the limited precision of the analogue signals.

9. An example: circuits for the velocity field computation

We will consider next some specific networks for solving the optical flow computation. The simpler case is when the measurements of the perpendicular component of the velocity, $v_i^\perp$, at n points along the contours, are exact. In this case, the discretized Euler–Lagrange equations, corresponding to the second regularization method, (8), are (Hildreth 1984*a*)

$$(2+\kappa_i^2)\,v_i^\top - v_{i+1}^\top - v_{i-1}^\top = d_i, \tag{13}$$

where κ_i^2 is the curvature of the curve at location i, d_i is a function of the data $v_i^\perp$ and the curve and $v_i^\top$ is the unknown tangential component of the velocity v_i at location i to be computed. Figure 3*a*, *b* shows two simple networks that solve (13), where one network is the dual of the other. The equation describing the ith node, in the case of figure 3*b*, is

$$(2g+g_i)\,V_i - g\,V_{i+1} - g\,V_{i-1} = I_i \tag{14}$$

where V_i is the voltage, corresponding to the unknown $v_i^\top$, and I_i the injected current at node i, corresponding to the measurement $v_i^\perp$. It is surprising that this implementation does not require negative resistances. When the constraints are satisfied only approximately, (9), the equations are

$$\left.\begin{aligned}(2+e_{xi}^2)\,V_{xi} - V_{xi+1} - V_{xi-1} + c_i\,V_{yi} &= d_{xi},\\ (2+e_{yi}^2)\,V_{yi} - V_{yi+1} - V_{yi-1} + c_i\,V_{xi} &= d_{yi}\end{aligned}\right\} \tag{15}$$

where e_{xi} and e_{yi} depend on the contour and V_{xi} and V_{yi} are the voltages corresponding to the x and y component of the unknown velocity at location i. The corresponding network is shown in figure 3*c*. The resistances c_i, however, can be either positive or negative, and may therefore require active components such as operational amplifiers. More precisely, physically realizable linear resistances, whether in electrical or in chemical systems, must dissipate energy, that is, they are constrained to the upper right and the lower left quadrant in the I–V plane and can thus only be positive. There are at least three options for implementing negative resistances by using basic circuit components. (i) The positive and negative resistances can be replaced in a purely resistive network by inductances and capacitances, with impedance $i\omega L$ and $-i/(\omega C)$ respectively. The network equations are then formulated in terms of the currents and voltages at the fixed frequency ω. (ii) The negative resistance can be implemented by the use of operational amplifiers or similar active circuit elements (see Jackson 1960). (iii) One may exploit the negative impedance regions in such nonlinear systems as the tunnel diode or a Hodgkin–Huxley like membrane.

We have devised similar analogue networks for solving other variational problems (Poggio & Koch 1984) arising from regularization analysis of several early vision problems such as edge detection (Poggio *et al.* 1985) and surface interpolation (Terzopoulos 1983). These networks are analogue solutions to certain kinds of spline interpolation and approximation problems. For instance, in the case of surface interpolation the analogue network solves the biharmonic equation which

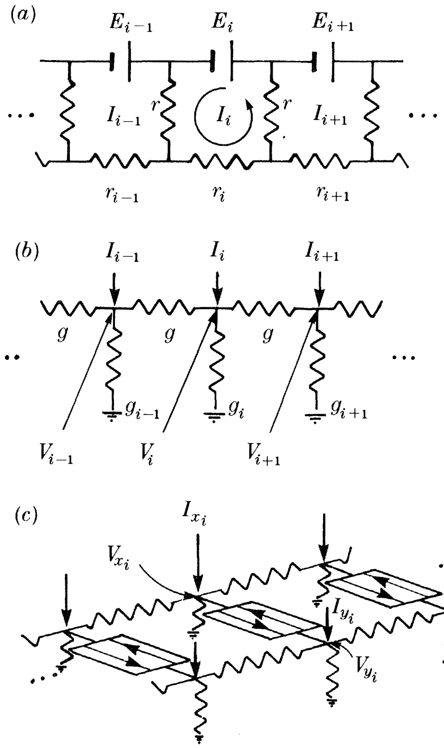

FIGURE 3. Three resistive networks computing the smoothest velocity field. The first two networks, duals of each other, correspond to the situation where the constraints imposed by the data are to be satisfied exactly. In (*a*) the equation for the current, which corresponds to the desired $v^{\top}$ in mesh i, is given by $(2r+r_i)I_i-rI_{i+1}-rI_{i-1}=E_i$, where the value of the battery E_i depends on the velocity data $v_i^{\perp}$, at location i. In (*b*) the voltage at node i, corresponding to $v_i^{\top}$, is given by $(2g+g_i)V_i-gV_{i+1}-gV_{i-1}=I_i$, where the injected current I_i depends on the velocity data. In this network sampling the voltage between nodes corresponds to linear interpolation between the node values. Network (*c*), consisting of two interconnected networks of the type shown in (*b*), solves the velocity field problem when the data are not exact. The equations for the ith nodes are $(2g_x+g_{xi})V_{xi}-g_x\,V_{xi+1}-g_x\,V_{xi-1}+c_i\,V_{yi}=d_{xi}$ and $(2g_y+g_{yi})\,V_{yi}-g_y\,V_{yi+1}-g_y\,V_{yi-1}+c_i\,V_{xi}=d_{yi}$. However, unlike the two purely passive networks shown above, an active element may be required, since the resistive cross-term c_i, relating the x and the y components of velocity, can be negative. Such a negative resistance can be mimicked, for instance, by operational amplifiers.

is the Euler–Lagrange equation corresponding to the variational problem associated with thin-plate splines. The stabilizing functionals used in regularization analysis of vision problems typically lead to local and limited connections between the components of the network.

Note that in general there is no unique network implementing a particular variational principle (witness the circuits in figure 3*a*, *b*, which are the dual of each other). For instance, graded networks of the type proposed by Hopfield in the context of associative memory (Hopfield 1984) can solve standard regularization principles, provided Hopfield's output function g_i, characterizing each neuron, is linear in V_i (for a further discussion see Koch *et al.* 1985).

10. Solving ill-posed problems with biological hardware

Analogue electrical networks are a natural hardware for computing the class of variational principles suggested by regularization analysis. Because of the well-known isomorphism between electrical and chemical networks (see, for instance, Busse & Hess 1973; Eigen 1974) that derives from the common underlying mathematical structure, appropriate sets of chemical reactions can be devised, at least in principle, to 'simulate' exactly the electrical circuits. Figures 4 and 6*a* show chemical networks that are equivalent (in the steady state) to the electrical circuit of figure 3*b*, *c*.

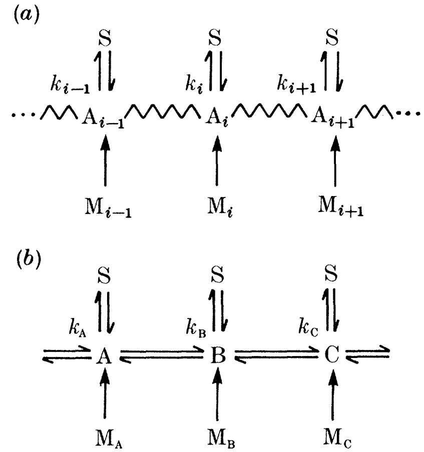

FIGURE 4. Two chemical networks computing the smoothest velocity field. Two examples of chemical networks solving the motion problem for exact measurements. They are equivalent, under steady-state conditions, to the electric circuit of figure 3*b*. (*a*) A diffusion–reaction system. A substance A (the concentration of which corresponds to the desired v^{T}) diffuses along a cable while reacting with an extracellular substance S (first order kinetics). The corresponding On-rate k_i varies from location to location. This could be achieved by a differential concentration of an enzyme catalysing the reaction or by varying the properties of the membrane where the reaction has to take place. The Off-rates can either be constant or vary with location. The inputs $v_i^{\perp}$ are given by the influxes of substance A. (*b*) A lumped chemical network, where *n* different, well-mixed substances, interact with each other and with the substrate S. Assuming first-order kinetics, these reactions can mimick a linear, positive resistance under steady-state conditions. The input is given by the influx M_X and the output by the concentration of X.

Electrical and chemical systems of this type therefore offer a computational model for early vision that is quite different from the digital computer. Equations are 'solved' in an implicit way, exploiting the physical constraints provided by Kirchhoff's laws. It is not difficult to imagine how this model of computation could be extended to mixed electrochemical systems by the use of transducers, such as chemical synapses, that can decouple two parts of a system, similarly to operational amplifiers (Poggio & Koch 1984).

Could neural hardware exploit this model of computation? Increasing evidence shows that electrotonic potentials play a primary role in many neurons (Schmitt *et al.* 1976) and that membrane properties such as resistance, capacitance and equivalent inductance (arising through voltage and time-dependent conductances;

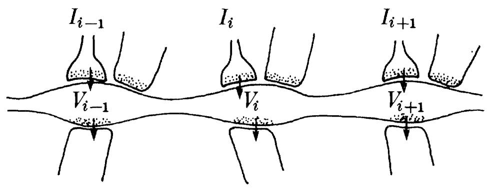

FIGURE 5. Neuronal network computing the smoothest velocity field. This schematic figure illustrates an hypothetical neuronal implementation of the regularization solution to the motion problem. A dendrite, acting both as pre- and post-synaptic element has a membrane resistance that can vary with location. It can implement under steady-state conditions the circuit 1*b*. The inputs, corresponding to the measurements $v^{\perp}$, are given by synaptically mediated current, while the output voltages, corresponding to the desired v^{T}, are sampled by dendro-dendritic synapses. The membrane resistance can be locally controlled by suitable synaptic inputs, corresponding to the curvature of the contour, from additional synapses that open channels with a reversal potential close to the resting potential of the dendrite.

see, for instance, Cole 1968; Koch 1984) may be effectively modulated by various types of neurotransmitters, acting over very different time scales (Marder 1984; Schmitt 1984). Dendrodendritic synapses and gap junctions serve to mediate graded, analogue interactions between neurons and do not rely on all-or-none action potentials (Graubard & Calvin 1976; Shepherd & Brayton 1979).

When implementing electrical networks in equivalent neuronal hardware, one can draw upon a large number of elementary circuit elements (for possible neuronal implementations see figures 5 and 6; for more details as regards the 'computational properties' of membranes, synapses and neurons see Koch & Poggio (1985)). Patches of neuronal membrane or cytoplasm can be treated as resistance and capacitance. Voltage sources may be mimicked by synapses on dendritic spines or thin dendrites (Koch & Poggio 1983), whereas synapses on large dendrites may

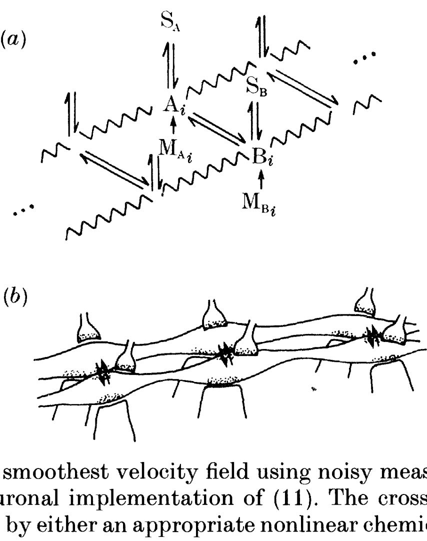

FIGURE 6. Computing the smoothest velocity field using noisy measurements. The figure shows a chemical and a neuronal implementation of (11). The cross-term c_i (15), which can be negative, is mimicked by either an appropriate nonlinear chemical reaction between the two substances A_i and B_i or by two reciprocal chemical synapses. If the corresponding cross-term in (11) is negative, the synapse must be inverting, presynaptic depolarization leading to a postsynaptic hyperpolarization.

act as current sources. Chemical synapses could effectively serve to decouple different parts of a network (see Poggio & Koch 1984). Chemical processes such as the reactions associated with postsynaptic effects or with neuropeptides could also be thought as part of a complex electrochemical network. Obviously, the analogy cannot be taken too literally. It would be very surprising to find the exact neural analogue of the circuit of figure 5 somewhere in the central nervous system. We are convinced, however, that the style of computation represented by analogue circuits represents a very useful model for neural computations as well as a challenge for future very large scale integration circuit designs.

11. Conclusion

The concept of ill-posed problems and the associated regularization theories seem to provide a satisfactory theoretical framework for much of early vision. This new perspective justifies the use of variational principles of a certain type for solving specific problems, and suggests how to approach other early vision problems. It provides a link between the computational (ill-posed) nature of the problems and the computational structure of the solution (as a variational principle). We have also discussed computational 'hardware' that is natural for solving variational problems of the type implied by standard regularization methods. The approach can be extended to other sensory modalities and to some motor control problems. Furthermore, the linear regularizing operators corresponding to standard, quadratic regularization principles can be synthetized in terms of simple associative learning schemes (Poggio & Hurlbert 1984) for which Kohonen (1984) has suggested simple neural implementations.

Despite its attractions, this theoretical synthesis of early vision also shows the limitations that are intrinsic to the variational solutions proposed so far, and in any case to the standard (Tikhonov's) forms of the regularization approach. The basic problem is the degree of smoothness required for the unknown function z that has to be recovered. If z is very smooth, then it will be robust against noise in the data, but it may be too smooth to be physically plausible. For instance, in visual surface interpolation, the degree of smoothness obtained from a specific form of (4) and (5), corresponding to so-called thin plate splines, smoothes depth discontinuities too much and often leads to unrealistic results (but see Terzopoulos 1984).

Different (for instance non-quadratic) variational principles may be used to attack the general problem of discontinuities (see Blake 1984*a*; Geman & Geman 1984; Marroquin 1984). Non-quadratic, that is non-standard, variational principles may also be needed to solve another fundamental problem in early vision, the problem of integrating different sources of information, such as stereo, motion, shape from shading, etc. This problem is ill-posed, not just because the solution is not unique (the normal case), but because the solution is usually over-constrained and may not exist (because of noise in the data). For instance, the problem of combining several different sources of surface information may easily lead to non-quadratic regularization expressions (though different 'non-interacting' constraints can be combined in a convex way, see Terzopoulos (1984)). These minimization problems will in general have multiple local minima.

The variational principles mostly considered for early vision processes, derived from standard regularization theory, are quadratic and therefore lead to linear networks. As we saw, this is not always to be expected. Though this is a topic that still needs to be explored, nonlinearities may greatly expand the rather restricted universe of computations that can be performed in terms of quadratic minimization principles. Again, analogue networks may be used to solve these minimization problems, that will in general have multiple local minima corresponding to the zeroes of the mixed potential (Brayton & Moser 1964; Oster *et al.* 1971). Schemes somewhat similar to stochastic minimization (Metropolis *et al.* 1953) or annealing (Kirkpatrick *et al.* 1983; Hinton & Sejnowski 1983) may be implemented by appropriate sources of gaussian noise driving the analogue network. The associated differential equation describing the dynamics of the system is then a stochastic differential equation.

Koch *et al.* (1985) show that nonlinear, graded networks like those used by Hopfield & Tank (1985) to compute next-to-optimal solutions of the travelling salesman problem (using a non-quadratic variational principle; see also Hopfield (1984)) may be used to solve the non-quadratic variational problem of reconstructing surfaces from sparsely sampled depth data in the presence of discontinuities like edges (Marroquin 1984). Needless to say, a number of biophysical mechanisms, such as somatic and dendritic action potentials, interactions between conductance changes, voltage and time-dependent conductances, neuropeptides, etc. (Koch & Poggio 1985), are likely to be used by neurons and patches of membrane to implement nonlinear operations.

The idea of vision problems as ill-posed problems originated from a conversation with Professor Mario Bertero of the University of Genoa about edge-detection work by V. Torre and T.P. (see Poggio & Torre 1984). We are grateful to Alan Yuille, Ellen Hildreth, Dimitri Terzopoulos, Jose Marroquin, Eric Grimson and Tom Collett for many discussions and comments and for critically reading the manuscript. We thank Linda Ardrey for drawing the figures and Carol Bonomo for typing the text. This report describes research done within the Artificial Intelligence Laboratory and the Center for Biological Information Processing (Whitaker College) at the Massachusetts Institute of Technology. Support for the A.I. Laboratory's research in artificial intelligence is provided in part by the Advanced Research Projects Agency of the Department of Defense under Office of Naval Research contract N00014-80-C-0505. The Center's support is provided in part by the Sloan Foundation and in part by Whitaker College.

Appendix 1

The main form of Hamilton's postulate of least action states that the motion of a dissipationless dynamical system, free from external disturbance, from a configuration at time t_1 to another configuration at time t_2, is such that the integral of its Lagrangian $L = T - U$ is stationary on the path followed. That is

$$\int_{t_1}^{t_2} L \, \mathrm{d}t = \text{extremum}. \tag{1.1}$$

If one extends Hamilton's postulate to dissipative systems acting under external forces it can be shown that

$$\int_{t_1}^{t_2} (L+W)\,\mathrm{d}t = \text{extremum} \tag{1.2}$$

must hold, where W is the virtual work done by the dissipative elements (for example resistances) and the sources.

There are two specialized variational principles for networks composed entirely of sources and linear or nonlinear dissipative elements. In such a network, the actual distribution of the currents will be such as to minimize the total content G of the system where G is given by

$$G = \int_0^I v\,\mathrm{d}i. \tag{1.3}$$

Conversely, the distribution of voltages will be such as to minimize the total co-content J of the system where

$$J = \int_0^V i\,\mathrm{d}v. \tag{1.4}$$

If the network considered contains only linear resistances and sources, then both of the above principles reduce to Maxwell's minimum heat theorem: the distribution of voltages and currents will be such as to minimize the total power dissipated as heat (MacFarlane 1970).

Appendix 2

Theorem. *Regularization principles of the form of* (4) *are equivalent to quadratic minimization problems in a Hilbert space. The latter have the form of the functionals minimized by networks of sources and resistances* (12).

Proof. We assume a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, which defines a quadratic norm $\|\cdot\|$. Equation (4) shows that an ill-posed problem can be formulated in terms of norms; that is, minimize

$$\|Az-y\|^2+\lambda\|Pz\|^2, \tag{2.1}$$

where P is any linear operator and y are the data. Writing this in terms of inner products, we have

$$\langle Az-y, Az-y\rangle+\lambda\langle Pz, Pz\rangle. \tag{2.2}$$

This is equal to

$$\lambda\langle Pz, Pz\rangle+\langle Az, Az\rangle-\langle Az, y\rangle-\langle y, Az\rangle+\langle y, y\rangle. \tag{2.3}$$

Since the last term is constant, it can be disregarded in the minimization. If the adjoints of the operators P and A are denoted by P^* and A^*, respectively, minimizing this expression is equivalent to minimizing

$$\lambda\langle z, P^*Pz\rangle+\langle z, A^*Az\rangle-2\langle z, A^*y\rangle. \tag{2.4}$$

Defining a new operator Q by $Q=\lambda P^*P+A^*A$, we can formulate the original variational problem as the problem of minimizing

$$\langle z, Qz\rangle-2\langle z, A^*y\rangle. \tag{2.5}$$

The first term can be identified with the total power dissipated in a linear resistive network, while the second term is the voltage- or current-source contribution. Note that Q is automatically self-adjoint. I_k and V_k in (12) are through and across variables, respectively, so that (12) can be put in the form of (2.5). Since Q is hermitian it can always be diagonalized. Equation (2.5) is then equivalent to (12) and our theorem is proved.

If Q is a linear positive definite and bounded operator, that is, it satisfies

$$m\langle z, z\rangle \leqslant \langle z, Qz\rangle \leqslant M\langle z, z\rangle, \tag{2.6}$$

for all $z \in H$, with H being a Hilbert space, and some $M, m > 0$, the vector z minimizing (2.5) is the unique solution of $Qz = b$ (thus the inner product $\langle z, Qz\rangle$ is H-elliptic and bounded, see Terzopoulos (1984)). The problem of minimizing the quadratic functional on a Hilbert space can be formulated as a Hilbert space minimum norm problem (Luenberger 1969).

Appendix 3

As shown in appendix 1, the content G of a network is stationary at the steady state. In particular, the proof holds if some resistances are negative (see, for instance, the proof based on Tellegen's theorem by Oster & Desoer (1971)). For linear constitutive relations (our case in this paper) the content is $G = P/2$, where P is the total dissipation. A resistive network with constant sources is always in the steady state, since it has no dynamic elements. Thus, the networks corresponding to the regularization principles find the correct solution, provided the variational principle (3) has a unique solution. A sufficient condition for the uniqueness of the variational solution is that the matrix Q is positive definite (see (2.6)).

It is also important to ask whether in practice these networks are stable. Even a purely resistive network will show some dynamics, because of small, unavoidable capacitances. In general, the solution is stable if all of the eigenvalues of the matrix Q are positive (see, for instance, Oster & Desoer 1971). An arbitrary resistive network with negative and positive resistances may have one or more negative eigenvalues and be therefore unstable. A network, however, representing a matrix Q with the properties of (2.6) is always stable, because Q is positive definite and has therefore only positive eigenvalues. In other words, conditions (easy to meet) that ensure uniqueness of the variational solution also ensure stability of the corresponding electrical network.

References

Barrow, H. G. & Tennenbaum, J. M. 1981 Interpreting line drawings as three dimensional surfaces. *Artif. Intelligence* **17**, 75–117.

Benett, M. V. L. 1972 A comparison of electrically and chemically mediated transmission. In *Structure and function of synapses* (ed. G. D. Pappas & D. P. Purpura), pp. 221–250. New York: Raven Press.

Bertero, M., Poggio, T. & Torre, V. 1986 Mathematical ill-posedness and regularization of vision problems. (In preparation.)

Blake, A. 1984*a* Parallel computation in low-level vision. PhD thesis, University of Edinburgh.

Blake, A. 1984*b* Reconstructing a visible surface. *Proc. natn. Conf. Artif. Intelligence*, pp. 23–26.

Brady, J. M., Grimson, W. E. L. & Langridge, D. J. 1980 Shape encoding and subjective contours. *First Conf. of Am. Ass. Artif. Intelligence*, pp. 15–17.

Brady, J. M. & Yuille, A. 1984 An extremum principle for shape from contour. *IEEE Trans. Pattern Analys. Machine Intelligence*, **6**, 288–301.

Brayton, R. K. & Moser, J. K. 1964 A theory of nonlinear networks. I. *Q. appl. Math.* **22**, 1–33.

Busse, H. & Hess, B. 1973 Information transmission in a diffusion-coupled oscillatory chemical system. *Nature, Lond.* **244**, 203–205.

Canny, J. F. 1983 Finding edges and lines in images. *Artif. Intelligence Lab. Tech. Rep.*, no. TR-720. M.I.T., Cambridge, Massachusetts.

Cole, K. S. 1968 *Membranes, ions and impulses.* Berkeley: University of California Press.

Eigen, M. 1974 Molecules, information, and memory: from molecules to neural networks. In *The neurosciences: third study program* (ed. F. O. Schmitt & F. G. Worden), pp. XIX–XXVII. Cambridge, Massachusetts: M.I.T. Press.

Ellias, S. A. & Stevens, J. K. 1972 The dendritic varicosity: a mechanism for electrically isolating the dendrites of cat retinal amacrine cells? *Brain Res.* **196**, 365–372.

Geman, S. & Geman, D. 1984 Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images. *IEEE Trans. Pattern Analys. Machine Intelligence* **6**, 721–741.

Gihman, I. I. & Skorohod, A. V. 1972 *Stochastic differential equations.* Berlin, Heidelberg, New York: Springer-Verlag.

Graubard, K. & Calvin, W. H. 1979 Presynaptic dendrites: implications of spikeless synaptic transmission and dendritic geometry. In *The neurosciences: fourth study program* (ed. F. O. Schmitt & F. G. Worden), pp. 317–332. Cambridge, Massachusetts: M.I.T. Press.

Grimson, W. E. L. 1981*a* A computer implementation of a theory of human stereo vision. *Phil. Trans. R. Soc. Lond.* B **292**, 217–253.

Grimson, W. E. L. 1981*b* *From images to surfaces: a computational study of the human early visual system.* Cambridge, Massachusetts: M.I.T. Press.

Grimson, W. E. L. 1982 A computational theory of visual surface interpolation. *Phil. Trans. R. Soc. Lond.* B **298**, 395–427.

Hadamard, J. 1923 *Lectures on the Cauchy problem in linear partial differential equations.* New Haven: Yale University Press.

Hildreth, E. C. 1984*a* *The measurement of visual motion.* Cambridge, Massachusetts: M.I.T. Press.

Hildreth, E. C. 1984*b* Computation of the velocity field. *Proc. R. Soc. Lond.* B **221**, 189–220.

Hinton, G. E. & Sejnowski, T. J. 1983 Optimal perceptual inference. *Proc. IEEE 1983 Conf. Computer Vision and Pattern Recognition*, Washington, D.C. pp. 448–453.

Hopfield, J. J. 1984 Neurons with graded response have collective computational properties like those of two-state neurons. *Proc. natn. Acad. Sci. U.S.A.* **81**, 3088–3092.

Hopfield, J. J. & Tank, D. W. 1985 Neural computation in optimization problems. *Biol. Cybern.* (In the press.)

Horn, B. K. P. 1974 Determining lightness from an image. *Computer Graphics Image Processing* **3**, 111–299.

Horn, B. K. P. & Schunck, B. G. 1981 Determining optical flow. *Artif. Intelligence* **17**, 185–203.

Ikeuchi, K. & Horn, B. K. P. 1981 Numerical shape from shading and occluding boundaries. *Artif. Intelligence* **17**, 141–184.

Jackson, D. S. 1960 *Analog computation.* New York: McGraw-Hill.

Kac, M. 1966 Can one hear the shape of a drum? *Am. math. Monthly* **73**, no. 4, part 2, 1–23.

Karplus, W. J. 1958 *Analog simulation: solution of field problems.* New York: McGraw-Hill.

Kass, M. H. 1984 Computating stereo correspondence. M.S. Thesis. M.I.T., Cambridge, Massachusetts.

Kirkpatrick, S., Gelatt, C. D., Jr. & Vecchi, M. P. 1983 Optimization by simulated annealing. *Science, Wash.* **220**, 671–680.

Koch, C. 1984 Cable theory in neurons with active, linearized membranes. *Biol. Cybern.* **50**, 15–33.

Koch, C., Marroquin, J. & Yuille, A. 1985 Analog 'neuronal' networks in early vision. *Artif. Intelligence Lab. Memo* no. 751. M.I.T., Cambridge, Massachusetts.

Koch, C. & Poggio, T. 1983 A theoretical analysis of electrical properties of spines. *Proc. R. Soc. Lond.* B **218**, 455–477.

Koch, C. & Poggio, T. 1985 Biophysics of computation: neurons, synapses and membranes. In *New insights into synaptic function* (ed. G. M. Edelman, W. E. Gall & W. M. Cowan). New York: Neurosciences Research Foundation and John Wiley.
Kohonen, T. 1984 *Self-organization and associative memory.* Heidelberg: Springer-Verlag.
Kron, G. 1945 Electric circuit models of the Schrodinger equation. *Phys. Rev.* **67**, 39–43.
Kuo, B. C. 1967 *Linear networks and systems.* New York: McGraw-Hill.
Luenberger, D. G. 1969 *Optimization by vector space methods.* New York: John Wiley.
MacFarlane, A. G. J. 1970 *Dynamical system models.* London: Georg Harrap Co.
MacNeal, R. H. 1951 The solution of elastic plate problems by electrical analogies. *J. appl. Mech.* **18**, 59–67. Also in *The bending and stretching of plates.* New York: Macmillan.
Marder, E. 1984 Mechanisms underlying neurotransmitter modulation of a neuronal circuit. *Trends Neurosci.* **7**, 48–53.
Marr, D. & Hildreth, E. C. 1980 Theory of edge detection. *Proc. R. Soc. Lond.* B **207**, 187–217.
Marr, D. & Poggio, T. 1976 Cooperative computation of stereo disparity. *Science, Wash.* **194**, 283–287.
Marr, D. & Poggio, T. 1979 A theory of human stereo vision. *Proc. R. Soc. Lond.* B **204**, 301–328.
Marroquin, J. 1984 Surface reconstruction preserving discontinuities. *Artif. Intelligence Lab. Memo*, no. 792. M.I.T. Cambridge, Massachusetts.
Mayhew, J. E. W. & Frisby, J. P. 1981 Psychophysical and computational studies towards a theory of human stereopsis. *Artif. Intelligence* **17**, 349–385.
McCulloch, W. S. & Pitts, W. 1943 A logical calculus of the ideas immanent in nervous activity. *Bull. math. Biophys.* **5**, 115–133.
Metropolis, N., Rosenbluth, A., Rosenbluth, M., Teller, A. & Teller, E. 1953 Equation of state calculations by fast computing machines. *J. chem. Phys.* **21**, 1087–1092.
Nashed, M. Z. 1974 Approximate regularized solutions to improperly posed linear integral and operator equations. In *Constructive and computational methods for differential and integral equations* (ed. G. Auger), pp. 289–296. Berlin: Akademie Verlag.
Nashed, M. Z. (ed.) 1976 *Generalized inverses and applications.* New York: Academic Press.
Nishihara, H. K. 1984 PRISM: a practical realtime imaging stereo matcher. *Artif. Intelligence Lab. Memo*, no. 780. M.I.T. Cambridge, Massachusetts.
Nishihara, H. K. & Poggio, T. 1984 Stereo vision for robotics. *Proc. first int. Symp. Robotics Res.* (In the press.)
Oster, G. F. & Desoer, C. A. 1971 Tellegen's theorem and thermodynamic inequalities. *J. theor. Biol.* **32**, 219–241.
Oster, G. F., Perelson, A. & Katchalsky, A. 1971 Network thermodynamics. *Nature, Lond.* **234**, 393–399.
Poggio, T. & Hurlbert, A. 1984 Associative learning of standard regularizing operators in early vision. *Artif. Intelligence Lab. Working Paper*, no. 264. M.I.T. Cambridge, Massachusetts.
Poggio, T. & Koch, C. 1984 An analog model of computation for the ill-posed problems of early vision. *Artif. Intelligence Lab. Memo*, no. 783. M.I.T. Cambridge, Massachusetts.
Poggio, T. & Torre, V. 1984 Ill-posed problems and regularization analysis in early vision. *Artif. Intelligence Lab. Memo*, no. 773. M.I.T. Cambridge, Massachusetts.
Poggio, T., Voorhees, H. & Yuille, A. 1985 Regularizing edge detection. *Artif. Intelligence Lab. Memo*, no. 776. M.I.T. Cambridge, Massachusetts.
Prigogine, I. 1967 *Thermodynamics of irreversible processes.* New York: Wiley-Interscience.
Reinsch, C. H. 1967 Smoothing by spline functions. *Numer. Math.* **10**, 177–183.
Schmitt, F. O. 1984 Molecular regulators of brain function: a new view. *Neuroscience* **13**, 991–1002.
Schmitt, F. O., Dev, P. & Smith, B. H. 1976 Electrotonic processing of information in brain cells. *Science, Wash.* **193**, 114–120.
Schunck, B. G. & Horn, B. K. P. 1981 Constraints on optical flow computation. *Proc. IEEE Conf. pattern recognition and image processing*, pp. 205–210.
Shepherd, G. M. & Brayton, R. K. 1979 Computer simulation of a dendrodendritic synaptic circuit for self- and lateral-inhibition in the olfactory bulb. *Brain Res.* **175**, 377–382.
Terzopoulos, D. 1983 Multilevel computational processes for visual surface reconstruction. *Computer Vision Graphics Image Processing* **24**, 52–96.
Terzopoulos, D. 1984 Multiresolution computation of visible-surface representations. Ph.D.

thesis, Department of Electrical Engineering and Computer Science, M.I.T., Cambridge, Massachusetts.

Terzopoulos, D. 1985 Integrating visual information from multiple sources for the cooperative computation of surface shape. In *From pixels to predicates: recent advances in computational and robotic vision* (ed. A. Pentland). New Jersey: Ablex. (In the press.)

Tikhonov, A. N. 1963 Solution of incorrectly formulated problems and the regularization method. *Soviet Math. Dokl.* **4**, 1035–1038.

Tikhonov, A. N. & Arsenin, V. Y. 1977 *Solutions of ill-posed problems.* Washington, D.C.: Winston & Sons.

Torre, V. & Poggio, T. 1985 On edge detection. *IEEE Computer Graphics Image Processing.* (In the press.)

Ullman, S. 1976 Filling in the gaps: the shape of subjective contours and a model for their generation. *Biol. Cybern.* **25**, 1–6.

Ullman, S. 1979*a* The interpretation of structure from motion. *Proc. R. Soc. Lond.* B **203**, 405–426.

Ullman, S. 1979*b* Relaxation and constrained optimization by local processes. *Computer Graphics Image Processing* **9**, 115–125.

Wahba, G. 1980 Ill-posed problems: numerical and statistical methods for mildly, moderately and severely ill-posed problems with noisy data. *Tech. Rep.* no. 595, University of Wisconsin, Madison.

Wax, N. 1954 *Noise and stochastic processes.* London: Dover Publications.

Yuille, A. 1983 The smoothest velocity field and token matching schemes. *Artif. Intelligence Lab. Memo*, no. 724. M.I.T. Cambridge, Massachusetts.