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Convolution and Correlation Algebras

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Abstract

An algebraic characterization of convolution and correlation is outlined. The basic algebraic structures generated on a suitable vector space by the two operations are described. The convolution induces an associative Abelian algebra over the real field; the correlation induces a not-associative, not-commutative – but Lie-admissible algebra – with a left unity. The algebraic connection between the two algebras is found to coincide with the relation of isotopy, an extension of the concept of equivalence. The interest of these algebraic structures with respect to information processing is discussed.

1. Introduction

As long as a system is linear and space- or time-invariant, its operation upon an incoming signal may be represented as the convolution of the signal with the particular response function (the Green function) characterizing the system.
Convolution is a concept of wide application; window, grating, filter, transfer function, dissipative response and feature extractor are all terms belonging to its language and are taken from such diverse fields as optics, electronic engineering, sensory physiology and pattern recognition, with physics providing many more examples.

The convolution operation relates the input $x(\zeta)$ to the output $y(\zeta)$ through the Green function of the system $h(\zeta)$, according to

$$y(\zeta)=\int x(\mu)\,h(\zeta-\mu)\,d\mu=x*h\,. \tag{1}$$

In practice, one is usually confronted with the problems of deconvolution – identifying $h(\zeta)$ from pairs of $x(\zeta)$ and $y(\zeta)$ – or deblurring – finding the $x(\zeta)$ associated with a given $y(\zeta)$ and $h(\zeta)$. In solving this class of problems an important and almost indispensible tool is the correlation operation, defined as

$$c(\zeta)=\int g(\mu)\,f(\mu+\zeta)\,d\mu=g\circledast f\,. \tag{2}$$

Correlation is no less restricted in range of application than convolution, and it has been increasingly recognized in recent years that the two concepts form a mathematical language which underlies many different formalisms in physics (see, for example Martin, 1968) as well as in the information sciences.

The single field of holography is an excellent illustration of the broad applicability of these concepts.

The transformation achieved in holography from the recording to the reconstruction can be generalized as a combination of convolution and correlation:

$$B'=(A\circledast B)*A'\,, \tag{3}$$

where A, B are signals, A' is the "recall" signal, B' is the reconstructed one. The same mathematical structure is basic to today's broad field of optical computing (Stroke, 1969, 1972; Toraldo di Francia, 1969), and in the communication sciences a variety of techniques (Leith, 1971) are recognized as holographic-like, sharing the same mathematical formulation of holography. Many processes we call holographic-like depend upon rather different physical mechanisms. Again it is the common underlying formal structure, Eq. (3), that unifies them. For instance the most meaningful aspect of the analogy between holography and memory brought up in recent years (Julesz, Pennington, 1965; Longuet-Higgins, 1968, 1970; van Heerden, 1970; Watson, 1971) is clearly not directly concerned with the physical mechanisms involved but rather with the underlying logic, a particular sequence of convolution and correlation (Borsellino, Poggio, 1971, 1972; Gabor, 1969; Poggio, 1970). A few functional properties of nervous systems support this point of view, suggesting the existence of a correlation principle of brain function (Reichardt, 1957, von Seelen, Reinig, 1972; Altes, 1971).

Therefore it seems quite interesting to consider the formal structures induced by the operations of correlation and convolution from an algebraic point of view. This approach might offer some insight into the general properties of those (information) systems characterized by the convolution and correlation structures.

In this paper, only the main ideas of an algebraic approach will be outlined. More specific results will be dealt with later on (Poggio, 1973b).

2. The Correlation and Convolution Algebras

Convolution and correlation operations are defined in the usual way on the linear space of functions with a "well-behaved" Fourier transform (for example the Hilbert space $L_2(-\infty, \infty)$ or suitable restrictions of it). The convolution function of $f(t)$ and $g(t)$ is then

$$(f * g)_\tau = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, g(\tau - t)\, dt \tag{4}$$

and the correlation is

$$(f \circledast g)_\tau = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, g(t+\tau)\, dt\,. \tag{5}$$

If f and g are aperiodic functions, definitions (4) and (5) take the forms $\left(\text{with } \int_{-\infty}^{\infty} |f(t)|\, dt,\ \int_{-\infty}^{\infty} |g(t)|\, dt \text{ finite}\right)$

$$(f * g)_\tau = \int_{-\infty}^{\infty} f(t)\, g(\tau - t)\, dt\,, \tag{6}$$

$$(f \circledast g)_\tau = \int_{-\infty}^{\infty} f(t)\, g(t+\tau)\, dt\,. \tag{7}$$

The extension to functions of many variables is straightforward.

The Fourier isophormic relationships are (disregarding normalization factors)

$$(f * g) \to F(\omega)\, G(\omega) \tag{8}$$

and

$$(f \circledast g) \to \bar{F}(\omega)\, G(\omega) \tag{9}$$

where F and G are the Fourier transforms of f and g, and $\bar{F}$ is the complex conjugate of F. Eq. (9) is usually referred to as the Wiener theorem.

It can be seen from definitions (4) and (5) that under the apparent similarity an important difference exists: convolution is commutative and associative, correlation is neither commutative nor associative. As a matter of fact, in the Fourier isomorphic space, where multiplication is by definition commutative and associative, the following relations hold

$$\begin{matrix} f * g \leftrightarrow FG \\ g * f \leftrightarrow GF \end{matrix} \; = \; \begin{matrix} f * (g * h) \leftrightarrow FGH \\ (f * g) * h \leftrightarrow FGH \end{matrix} \; = \tag{10}$$

$$\begin{matrix} f \circledast g \leftrightarrow \bar{F}G \\ g \circledast f \leftrightarrow \bar{G}F \end{matrix} \; \neq \; \begin{matrix} f \circledast (g \circledast h) \leftrightarrow \bar{F}\bar{G}H \\ (f \circledast g) \circledast h \leftrightarrow F\bar{G}H \end{matrix} \; \neq \tag{11}$$

It is straightforward to check (see Appendix I) that, with the usual operations (addition and multiplication with scalars) correlation and convolution induce, on the space L_2, two algebras, denoted respectively by U^* and $U^{\circledast}$. We may regard the algebras U^* and $U^{\circledast}$ as having the same quantities but different laws for forming products.

In the following, we restrict our attention to discrete spaces. In particular, we consider the linear algebras B^* and $B^{\circledast}$ over the real field, which consist of all vectors f

$$f = (\ldots, f_{-1}, f_0, f_1, \ldots) \tag{12}$$

together with the following product laws

$$(f * g)_m = \sum_i f_i g_{m-i}\,, \tag{13}$$

$$(f \circledast g)_m = \sum_i f_i g_{m+i}\,. \tag{14}$$

As shown in Appendix II the algebras B^* and $B^{\circledast}$ have a direct physical interpretation. In fact the infinite vectors f can be assumed to represent in terms of the Whittaker basis the class of band-limited real-valued signals.

3. Structure of B^* and $B^{\circledast}$

We begin by outlining the structure of the algebras B^* and $B^{\circledast}$ over the real field $\mathbb{R}$. They consist of the linear space L of all vectors (12) together with a set of quantities γ_{ijk} in $\mathbb{R}$ such that the products

$$\begin{aligned} v = f * g &= (\ldots, f_{-1}, f_0, f_1, \ldots) * (\ldots, g_{-1}, g_0, g_1, \ldots) \\ &= (\ldots v_{-1}, v_0, v_1, \ldots) \end{aligned} \tag{15}$$

and

$$\begin{aligned} w = f \circledast g &= (\ldots, f_{-1}, f_0, f_1, \ldots) \circledast (\ldots, g_{-1}, g_0, g_1, \ldots) \\ &= (\ldots w_{-1}, w_0, w_1, \ldots) \end{aligned} \tag{16}$$

are defined – in B^* and $B^{\circledast}$ respectively – by

$$v_k = \sum_{ij} \gamma^*_{ijk} f_i g_j \tag{17}$$

and by

$$w_k = \sum_{ij} \gamma^{\circledast}_{ijk} f_i g_j\,. \tag{18}$$

From definitions (13) and (14) one obtains

$$\gamma^*_{ijk} = \delta_{k,j+i}\,, \tag{19}$$

$$\gamma^{\circledast}_{ijk} = \delta_{k,j-i}\,, \tag{20}$$

where $\delta_{t,l}$ is the usual Kronecker symbol. The multiplication constants γ_{ijk} define completely the structure of the convolution and correlation algebras.

As a "measure" of the commutativity and associativity of the two algebras we introduce the *commutators*

$$\begin{aligned}[f,g]^* &= f * g - g * f \\ [f,g]^{\circledast} &= f \circledast g - g \circledast f\end{aligned} \tag{21}$$

and the *associators*

$$\begin{aligned}(f,g,h)^* &= f * (g * h) - (f * g) * h \\ (f,g,h)^{\circledast} &= f \circledast (g \circledast h) - (f \circledast g) \circledast h.\end{aligned} \tag{22}$$

It is again straightforward to show that for B^*

$$[f,g]^* \equiv 0 \qquad (f,g,h)^* \equiv 0\,, \tag{23}$$

but for $B^{\circledast}$

$$[f,g]^{\circledast} \not\equiv 0 \qquad (f,g,h)^{\circledast} \not\equiv 0\,, \tag{24}$$

which again characterizes B^* as an Abelian, associative algebra and $B^{\circledast}$ as a not-associative, not-commutative one (see Appendix III).

The general problem of non-associative algebras has not been studied very much: we will refer mainly to the work of Albert (Albert, 1942; see also Schäfer, 1966). On the other hand theories of special not-associative algebras have already yielded much of importance. Among them we especially mention the alternative algebras, Lie algebras and Jordan algebras. It is perhaps not without interest that the not-associative algebras mentioned above are basic tools in modern physics. The correlation algebra does not belong to any one of these special cases (see Table 1), as is easily checked. On the other hand the algebra generated by the correlation commutator is a Lie algebra, that is

$$\begin{aligned}&[f,f]^{\circledast} \equiv 0\,, \\ &[[f,g]^{\circledast},h]^{\circledast} + [[g,h]^{\circledast},f]^{\circledast} + [[h,f]^{\circledast} g]^{\circledast} \equiv 0\,.\end{aligned} \tag{25}$$

The Jacobi relation (25) represents a property of associative algebras, which, however, does not hold in general for not-associative algebras. Since any algebra is defined as Lie-admissible if its commutator algebra is a Lie algebra (Santilli, 1968), we have then the following non-trivial result, namely that *the correlation algebra is Lie-admissible* [1]. A very interesting feature of Lie-admissible algebras – which is likely to have a direct physical meaning (Santilli, 1968) – is that methodological procedures used in the theory of Lie-algebras can be extended to Lie-admissible algebras if some other suitable condition is introduced.

Table 1. Characteristic properties of some not-associative algebras

Alternative algebras	$x^2 y \equiv x(xy)$ $xy^2 \equiv (xy)\,y$
Lie algebras	$x^2 \equiv 0$ $((xy)\,z) + ((yz\,x) + ((zx)\,y) \equiv 0$
Jordan algebras	$[x,y] = 0$ $(xy)\,x^2 = x(yx^2)$
Correlation algebra	$x^2 y \equiv (yx)\,x$ $xy^2 \equiv y(xy)$

4. The Multiplication Spaces of B^* and $B^{\circledast}$

To characterize from an algebraic point of view the properties of the not-associative algebra $B^{\circledast}$, we now introduce – following Albert's treatment – the multiplication spaces of the convolution and correlation algebras.

We define $\Gamma^{(j)}$ as the square matrix with γ_{ijk} in the i^{th} row and k^{th} column, and then we associate with every $f \in B^{\cdot}$ (where $\cdot$ indicates $\circledast$ or $*$) the following matrix

$$\Gamma_f = \cdots + \Gamma^{(-1)} f_{-1} + \Gamma^{(0)} f_0 + \Gamma^{(1)} f_1 + \cdots. \tag{26}$$

It is then clear that

$$g \cdot f = g\Gamma_f\,, \tag{27}$$

where $g\Gamma_f$ is computed as usual. In an analogous way we define the matrix $\Delta^{(i)}$ with element $(j,k) = \gamma_{ijk}$ and we associate with every $f \in B^{\cdot}$ the matrix

$$\Delta_f = \cdots + \Delta^{(-1)} f_{-1} + \Delta^{(0)} f_0 + \Delta^{(1)} f_1 + \cdots. \tag{28}$$

It is again clear that

$$f \cdot g = g\Delta_f\,. \tag{29}$$

Since matrix multiplication is associative,

$$(h \cdot f) \cdot g = h(\Gamma_f \Gamma_g) = (h\Gamma_f)\,\Gamma_g\,. \tag{30}$$

All linear transformations R on L with matrices Γ_f (defined by $h \to hR_f = h \cdot f$) form the right multiplication space $R(B^{\cdot})$ of $B^{\cdot}$. The linear transformation L_f given by $h \to f \cdot h = hL_f$, forms the left multiplication space $L(B^{\cdot})$. The linear mapping $f \to L_f$ determines and is completely determined by $R(B^{\cdot})$. For correlation and convolution algebras, Γ_f and Δ_f have the following matrix form (when L is the vector space

[1] An algebra is said to be Jordan-admissible if the anticommutator $\{a,b\} = ab + ba$ is a Jordan algebra. It is easy to prove that $B^{\circledast}$ is not Jordan-admissible.

representing band-limited functions in the $\{a_n\}$ basis):

$$\Gamma_f^* = \Delta_f^* = \begin{pmatrix} & & \cdots & & \\ \cdots & f_0 & f_1 & f_2 & \cdots \\ \cdots & f_{-1} & f_0 & f_1 & \cdots \\ \cdots & f_{-2} & f_{-1} & f_0 & \cdots \\ & & \cdots & & \end{pmatrix}, \quad (31)$$

$$\Gamma_f^{\circledast} = \begin{pmatrix} & & \cdots & & \\ \cdots & f_{-2} & f_{-1} & f_0 & \cdots \\ \cdots & f_{-1} & f_0 & f_1 & \cdots \\ \cdots & f_0 & f_1 & f_2 & \cdots \\ & & \cdots & & \end{pmatrix}, \quad (32)$$

$$\Delta_f^{\circledast} = \begin{pmatrix} & & \cdots & & \\ \cdots & f_0 & f_{-1} & f_{-2} & \cdots \\ \cdots & f_1 & f_0 & f_{-1} & \cdots \\ \cdots & f_2 & f_1 & f_0 & \cdots \\ & & \cdots & & \end{pmatrix}. \quad (33)$$

It is instructive that for real-valued functions belonging to TL_2 (time-limited functions), the Γ and Δ matrices take, in the discrete Fourier basis, the following form with

$$F = (\ldots, C_{-1}, C_0, C_1, \ldots), \quad (34)$$

$$\Gamma_F^* = \Delta_F^* = \begin{pmatrix} & & \cdots & & \\ \cdots & C_{-1} & 0 & \cdots & \\ \cdots & 0 & C_0 & 0 & \cdots \\ & \cdots & 0 & C_1 & \cdots \\ & & \cdots & & \end{pmatrix}, \quad (35)$$

$$\Gamma_F^{\circledast} = \begin{pmatrix} & & \cdots & & \\ & \cdots & 0 & C_1 & \cdots \\ \cdots & 0 & C_0 & 0 & \cdots \\ \cdots & C_{-1} & 0 & \cdots & \\ & & \cdots & & \end{pmatrix}, \quad (36)$$

$$\Delta_F^{\circledast} = \begin{pmatrix} & & \cdots & & \\ \cdots & C_1 & 0 & \cdots & \\ \cdots & 0 & C_0 & 0 & \cdots \\ & \cdots & 0 & C_{-1} & \cdots \\ & & \cdots & & \end{pmatrix}. \quad (37)$$

It is easy to check the specific form of $\Gamma^{(i)}$ and $\Delta^{(i)}$, as well as the equivalence of definitions (13), (14) with definitions

$$f * g = f\Gamma_g^* = g\Delta_f^*, \quad (38)$$

$$f \circledast g = f\Gamma_g^{\circledast} = g\Delta_f^{\circledast}. \quad (39)$$

We have thus related the properties of B^* and $B^{\circledast}$ to the properties of the corresponding spaces $R(B)$ and $L(B)$ in which multiplication does satisfy the associative law. The reason for carrying out this treatment for B^* as well as for $B^{\circledast}$ will become clear later when we will find the *algebraic* connection between convolution and correlation. The matrices Δ and Γ represent also a specific algorithm for the computation of convolution and correlation.

5. Subalgebras and Ideals

A quite obvious subalgebra of B^* as well as of $B^{\circledast}$ is the set of all even vectors, that is the vectors p:

$$p_n = p_{-n}. \quad (40)$$

We call these subalgebras E^* and $E^{\circledast}$ respectively. It turns out that $E^{\circledast}$ is associative and commutative; actually $E^* \equiv E^{\circledast}$. The set of odd vectors is a subalgebra neither of $B^{\circledast}$ nor of B^*. On the other hand, the commutator correlation algebra has the interesting multiplication diagram (with respect to even and odd vectors) which is shown in Table 2, together with the

Table 2

$\times$	P	D		$\times^\dagger$	P	D
P	0	D		P	P	0
D	D	0		D	0	P

Symbols P and D denote even and odd functions respectively. The products are defined as follows:

$$a \times b = [a, b]^{\circledast} = a \circledast b - b \circledast a,$$
$$a \times^\dagger b = \{a, b\}^{\circledast} = a \circledast b + b \circledast a.$$

The functions (or the representing vectors) are assumed to be *real*-valued.

dual diagram pertaining to the anticommutator. Therefore the $[\,,]^{\circledast}$ operation is able to "extract" from two real signals the common "antisymmetry" or "oddness"; the dual statement is true for $\{\,,\}^{\circledast}$. The meaning of such a "symmetry extraction" may be important in problems of information processing (see Reichardt, 1973).

Interesting subalgebras of U^* and $U^{\circledast}$ are the functions whose frequency transform is identically zero outside some interval belonging to $|\omega_0|$. It is always possible to select from them a set of invariant subalgebras Π_α for which the following properties hold

$$\begin{aligned} &\Pi_\alpha \cdot U^{\cdot} \subseteqq \Pi_\alpha \\ &\Pi_\alpha \cap \Pi_{\alpha'} \equiv 0 \quad \text{where } U^{\cdot} \text{ is } U^* \text{ or } U^{\circledast} \\ &\cup \Pi_\alpha = U^{\cdot}. \end{aligned} \tag{41}$$

The Π_α's are of course ideals of U^* and $U^{\circledast}$. According to the usual decomposition theorems, U^* and $U^{\circledast}$ are reducible to the sum of a suitable set of Π_α. The properties of U^* and $U^{\circledast}$ (and of course of the associated discrete algebras B^* and $B^{\circledast}$) are therefore the sum of the properties of Π_α's, and it is possible to perform multiplication and addition componentwise. For example, for $f, g \in U^{\circledast}$

$$\begin{matrix} f = h_1 + h_2 \\ g = l_1 + l_2 \end{matrix} \quad \text{with} \quad \begin{matrix} h_1, l_1 \in \Pi_1 \\ h_2, l_2 \in \Pi_2 \end{matrix}, \tag{42}$$

it holds that

$$f \circledast g = h_1 \circledast l_1 + h_2 \circledast l_2 . \tag{43}$$

In the limit of Π_α being single-frequency components, the decomposition theorem stated above becomes the Wiener theorem. It is perhaps interesting to notice that the analysis of a signal in terms of ideals which are not the single Fourier eigenfunctions is nevertheless an "orthogonal" decomposition with respect to correlation and convolution.

Another interesting and obvious result is that the set of all "noiselike unities", that is, the vectors $\boldsymbol{n}^\alpha$ such that

$$(\boldsymbol{n}^\alpha * \boldsymbol{n}^{\alpha'})_i = \delta_{\alpha,\alpha'} \delta_{0,i} , \tag{44}$$

generate a subalgebra $N^{\circledast}$ of $B^{\circledast}$ (see Appendix IV). It may be recognized that the associated functions play an important role in holographic associative memories and in holographic-like analogues of human memory (Longuet-Higgins, 1968; Gabor, 1969; Borsellino, Poggio, 1972). All complicated patterns (practical examples are printed letters or ground glass surfaces or impulse sequences which are coded according to pseudorandom shift register codes) are, in first approximation, noiselike functions (Gabor, 1969) whose *algebraic* characterization is given by (44). We will return to this point later.

6. Divisors of Zero and Simpleness

We shall call the element c belonging to an algebra U a *right divisor of zero if* there exists $b \neq 0$ such that $b \cdot c = 0$ and $c \neq 0$. A quantity c of U is called an *absolute divisor of zero* if $c \neq 0$ and $c \cdot b = 0$ for every $b \in U$.

According to these customary definitions it is obvious that B^* and $B^{\circledast}$ do not have absolute divisors of zero. *In general* they have divisors of zero which correspond to the functions ($\neq 0$) whose spectrum is zero in some interval inside (ω_0). It is therefore clear that B^* and $B^{\circledast}$ are *not*, in general, simple algebras. An algebra U over $\mathscr{I}$ is called *simple* when zero and U are the only ideals of U. Moreover, every division algebra is simple. It will sometimes be useful to consider the subalgebras D^* and $D^{\circledast}$ of B^* and $B^{\circledast}$ as defined as the set of all vectors corresponding to the bandlimited functions with a non-zero spectrum in *all* the intervals $|\omega_0| \cdot D^*$ and $D^{\circledast}$ are simple division algebras without ideals: in particular $D^{\circledast}$ contains $N^{\circledast}$. Some implications of this algebraic picture will be discussed later.

7. Unity

The unity of B^* is shown to be

$$e = (\ldots 0, 0, 1, 0, 0 \ldots) . \tag{45}$$

The not-associative algebra $B^{\circledast}$ has only the left unity

$$e = (\ldots 0, 0, 1, 0, 0 \ldots) \tag{46}$$

which is identical with (45). Therefore

$$\Delta_e^* = \Gamma_e^* = \begin{pmatrix} & & \cdots & & & \\ \cdots & 0 & 1 & 0 & \cdots & & \\ & \cdots & 0 & 1 & 0 & \cdots & \\ & & \cdots & 0 & 1 & 0 & \cdots \\ & & & \cdots & & & \end{pmatrix}, \tag{47}$$

$$\Delta_e^{\circledast} = \begin{pmatrix} & & \cdots & & & \\ \cdots & 0 & 1 & 0 & \cdots & & \\ & \cdots & 0 & 1 & 0 & \cdots & \\ & & \cdots & 0 & 1 & 0 & \cdots \\ & & & \cdots & & & \end{pmatrix}, \tag{48}$$

but

$$\Gamma_e^{\circledast} = \begin{pmatrix} & & & \cdots & & & \\ & & \cdots & 0 & 1 & 0 & \cdots \\ & \cdots & 0 & 1 & 0 & \cdots & \\ \cdots & 0 & 1 & 0 & \cdots & & \\ & & \cdots & & & & \end{pmatrix}. \tag{49}$$

We notice that the quantity e represents, in the Whittaker basis, the bandlimited function

$$a_0(t) = \frac{\sin \omega_0 t}{\omega_0 t} \tag{50}$$

whose Fourier transform is

$$\mathscr{I}[a_0(t)] = \begin{cases} 1 & \text{for} \quad |\omega| < \omega_0 \\ 0 & \text{for} \quad |\omega| \geqq \omega_0 . \end{cases} \tag{51}$$

8. Isotopy

Up to now we have characterized to some extent the convolution and the correlation algebras. B^* turned out to be an associative, commutative algebra with unity, generally not simple. We stated that $B^{\circledast}$ is a not-associative, not-commutative algebra with a left unity, generally not simple, with the same ideals as B^* and a commutator algebra which is a Lie algebra so that $B^{\circledast}$ is also Lie-admissible. We have not yet stated the algebraic connection between convolution and correlation. For this purpose we shall introduce in the following the concept of *isotopy* of algebras, an extension of the concept of equivalence (Albert, 1942). In this way we will be able to overcome the undesirable narrowness of the concept of equivalence for non-associative algebras, and then to obtain an associative "representation" of the correlation algebra.

As we stated before, if two algebras A and A' are given on a linear space L, the multiplication spaces $R(A)$ and $R(A')$ with the corresponding mappings

$$a \to R_a$$

and

$$a \to R'_a$$

are therefore determined. We now say that A is isotopic to A' if there exist non-singular linear transformations P, Q, C such that

$$R'_f = P R_{fQ} C \,. \tag{52}$$

It is possible to show that conditions (52) are equivalent to

$$L'_f = Q L_{fP} C \,. \tag{53}$$

The relation of equivalence is a particular case of the relation of isotopy.

It is generally more convenient to replace A' by an equivalent algebra so that we can introduce the concept of a *principal* isotope of an algebra to which every isotope of the algebra is equivalent. We shall say that A is a *principal isotope* of A_0 if there exist non-singular linear transformations P and Q such that

$$R_f = P R_{fQ} \,, \quad L f = Q L_{fP} \,. \tag{54}$$

The relation of principal isotopy – as well as the relation of isotopy – is a formal equivalence relation.

With respect to correlation algebra the following result holds (Borsellino, Poggio, 1971):

the correlation and convolution algebras $B^{\circledast}$ and B^* are principal isotopes.

The proof is immediate when it is noticed that the relation of isotopy does not depend upon the basis chosen. Therefore we merely assume for $R_f^{\circledast}, L_f^{\circledast}$ and R_f^* the matrix representations (32), (33), (31). It is then easy to check that relations (54) are satisfied for B^* and $B^{\circledast}$ (being A_0 and A) by the following non-singular matrices Q, P

$$Q = \begin{pmatrix} & & \cdots & & \\ \cdots & 1 & 0 & \cdots & \\ \cdots & 0 & 1 & 0 & \cdots \\ & \cdots & 0 & 1 & \cdots \\ & & \cdots & & \end{pmatrix}, \tag{55}$$

$$P = \begin{pmatrix} & & & \cdots & \\ & \cdots & 0 & 1 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 1 & 0 & \cdots & \\ & \cdots & & & \end{pmatrix}. \tag{56}$$

The matrices Q and P represent the identity and the reflection transformations. They satisfy the conditions

$$\Gamma_f^{\circledast} = P \Gamma_{fQ}^* \,, \tag{57}$$

$$\Delta_f^{\circledast} = Q \Delta_{fP}^* \,. \tag{58}$$

The same matrices define the same relation of isotopy for the vectors in the Fourier space associated with timelimited real-valued functions.

It is important to stress again that the concept of isotopy coincides with the concept of equivalence for associative algebras with unity. In other words, the correlation algebra has some associative "representation" through the isotopic relations (57) and (58). By means of them, a few theorems (Albert, 1942; Bruck, 1944) become readily available. They allow one to derive properties of $B^{\circledast}$ from the structure of B^* and to characterize further the correlation algebra (Poggio, 1973b). For example, interesting statements can be proved about alternative isotopes of $B^{\circledast}$; the center and the centralizer of $B^{\circledast}$ have a non-trivial structure (see Appendix V); the properties which characterize the correlation algebra can be compared with the ones pertaining to other not-associative algebras (Appendix VI).

9. Concluding Comments

We shall now briefly mention some implications of the algebraic properties which have been outlined above.

As we already said in the introduction, a very large body of optical techniques – and electronic ones – show the structure of a convolution-correlation algebra. It is then rather easy to translate physical concepts into algebraic terms. For example, the concept of frequency filtering through convolution and correlation must be related to the decomposition in ideals of U^* and $U^{\circledast}$. Information processing by means of frequency "channels", that is by operating independently on different regions of the frequency band, can show the formal structure of a *non-simple* convolution-correlation algebra. On the other hand, an associative holographic-like memory shall present, in general, the structure of a *simple* division algebra (no frequency "holes"). Threshold systems might allow the transition from a division algebra to a *non-simple* one; that is, from an organized, possibly hierarchical "filtering" stage to an associative one. Algebraic characterization of many other techniques in information processing is of course easily possible. At this point, however, we shall briefly discuss the special interest of the convolution-correlation algebra with respect to the theories of holographic-like associative recall. Associative holographic-like memories can be described with the following symbolism (Gabor, 1969). In order to associate a temporal or spatial signal a with b, we first store the function

$$\psi_{ab} = a * b \,. \tag{59}$$

To evoke a by means of b it is sufficient to correlate ψ_{ab} with b or a part of it, provided that b is noiselike (i.e., is a function with a delta-peaked autocorrelation). Then a is recalled according to

$$b \circledast \psi_{ab} = a * (b \circledast b) \simeq a \,. \tag{60}$$

In terms of the algebraic algorithms developed in relation with the multiplication spaces of B^* and $B^{\circledast}$, Eq. (60) can also be written as

$$b \circledast (b * a) = a \Delta_b^* \Delta_b^{\circledast} \,, \tag{61}$$

where $\Delta_b^* \Delta_b^{\circledast}$ is given by

$$\Delta_b^* \Delta_b^{\circledast} = \begin{pmatrix} & & \cdots & & \\ \cdots & b_0 & b_1 & \cdots & \\ \cdots & b_{-1} & b_0 & b_1 & \cdots \\ & \cdots & b_{-1} & b_0 & \cdots \\ & & \cdots & & \end{pmatrix} \begin{pmatrix} & & \cdots & & \\ \cdots & b_0 & b_{-1} & \cdots & \\ \cdots & b_1 & b_0 & b_{-1} & \cdots \\ & \cdots & b_1 & b_0 & \cdots \\ & & \cdots & & \end{pmatrix} . \tag{62}$$

Since the vector b is supposed to belong to $N^{\circledast}$, it is easy to check that only diagonal terms of the resulting matrix are different from zero, that is

$$\Delta_b^* \Delta_b^{\circledast} = \begin{pmatrix} & & \cdots & & \\ \cdots & 1 & 0 & \cdots & \\ \cdots & 0 & 1 & 0 & \cdots \\ & \cdots & 0 & 1 & \cdots \\ & & \cdots & & \end{pmatrix} = \Gamma_e^{\circledast} \text{ (say)} \tag{63}$$

giving, therefore, the result of Eq. (60).

It is interesting that this association scheme is formally equivalent to classical Pavlovian stimulus-response learning (Poggio, 1970). The formal organization of such a memory can be embedded in a vector space with the two superimposed structures of a correlation and a convolution algebra. In this language the basic characterization is that the key functions, or stimuli, (like b) must belong to $N^{\circledast}$, the subalgebra defined in an earlier paragraph. Since the set of "noiselike" functions is closed (see Appendix IV), it is of course not possible to have a noise-like basis for $U^{\circledast}$, that means for the complete signal (or stimuli) space. Rather, in order to implement an associative memory in a convolution-correlation structure, it seems necessary to induce a suitable isomorphism between the signal space and a noiselike set. That means some "*noise coding*" of the input signals which have to be mapped into random or pseudorandom sequences. It is not the purpose of this paper to discuss any potential interest of this observation with respect to neural codes and internal "language" of the brain.

Starting from the associative convolution-correlation property, many implementations are of course possible. As a typical example, a system incorporating a prefilter or coding stage and a holographic-like memory of the type (59), (60), will belong to the class of two-layer perceptrons (in the wide sense). In addition to an optimal noiselike coding of the input patterns according to their "survival value" simple non-linear "clean-up" procedures at the output can easily improve the signal-to-noise ratio of such an associative memory. An almost infinite number of more complex schemes are also possible (Poggio, 1970) – always presenting the basic holographic-like structure. The relationship between convolution and correlation as shown by the isotopic theorem can be expressed in words in the following way: correlation always implies a "direction" in the processing of information since it discriminates between the "order" of interacting signals and the "channels" from where they are coming; convolution does not. The introduction of the correlation algebraic structure besides the convolution algebra seems sufficient to provide simple associative memory and pattern recognition properties. On the other hand, not-associative algebraic structures may be basic in describing complex phenomena. Genetic algebras are an example. Moreover, it is possible to prove that a formal algebraic structure with two product laws, is able to mimic an associative Pavlovian learning if at least one of them is not-associative (Poggio, 1973b).

Finally, we want to suggest that correlation can be regarded, in a useful sense, as some first approximation to non-linear interactions of information flows. This point of view, which has some statistical counterparts, should characterize better the meaning and the range of application of the "correlation" approach.

In this sense it is clearly possible to extend both convolution and correlation as well as the basic holographic associative scheme [Eq. (60)] in order to take into account higher order non-linearities. From this point of view, for instance, holographic memories become a special case of a more general class of associative memories, based upon a non-linear extension of the convolution operation (Poggio, 1973a). Clearly, an approach like the algebraic one might be very appropriate for this kind of extension because of the usual difficulties analytical techniques are confronted with in obtaining general results for non-linear problems.

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Appendix I

We assume that the functions $f(t)$ are absolutely integrable in $(-\infty, +\infty)$, that is

$$\int_{-\infty}^{\infty} |f(x)|\, dx\, \exists\,. \tag{1.1}$$

This assumption is certainly too restrictive in many cases, but it is a convenient starting point. From (1.1) we are able to define a precise frequency-transform pair

$$F(\omega)=\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx\,. \tag{1.2}$$

We want to show that with the products defined by (4) and (5) the class of real-valued functions satisfying (1.1) becomes an algebra U^* and an algebra $U^{\circledast}$ over the real field.

Since the class of functions (1.1) is a Hilbert space, we need only show that:

a) the products (4), (5) are inner laws.

In fact, through the Wiener theorem, we can assume that if f and g have a Fourier transform their correlation and their convolutions also have a Fourier transform.

b) The following distributive laws hold:

$$\begin{aligned}(f+g)\circledast h&=f\circledast h+g\circledast h\\ h\circledast(f+g)&=h\circledast f+h\circledast g\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}(f+g)*h&=f*h+g*h\\ h*(f+g)&=h*f+h*g\,.\end{aligned} \tag{1.4}$$

In fact they are trivially true because of the distributivity of integrals.

c) The multiplications are bilinear:

$$\begin{aligned}\alpha(f\circledast g)&=(\alpha f)\circledast g=f\circledast(\alpha g)\\ \alpha(f*g)&=(\alpha f)*g=f*(\alpha a)\,,\end{aligned} \tag{1.5}$$

where α is in the real field. This is again trivially true. Therefore $U^{\circledast}$ and U^* are two algebras according to a definition of algebra which does not assume the associative property for multiplication.

Appendix II

We consider the class of band-limited functions $|\omega_0|$ which are square-integrable. A function $f(t)\in L^2$ is said to be band-limited to ω_0 if $F(\omega)$ vanishes outside $(-\omega_0,\omega_0)$. We shall denote this class of functions as $BL^2_{\omega_0}$. The restriction, even if not completely necessary, is not without interest provided that all "physical" signals belong, for all practical purposes, to $BL^2_{\omega_0}$. The linear class of functions band-limited to $(-\omega_0,\omega_0)$ is closed and forms a Hilbert space all by itself. In terms of the Whittaker basis

$$a_n(t)=\frac{\sin(\omega_0 t-n\pi)}{\omega_0 t-n\pi}\,, \tag{2.1}$$

the functions $f(t)\in BL^2_{\omega_0}$ have the following representation

$$f(t)=\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\omega_0}\right)\frac{\sin(\omega_0 t-n\pi)}{\omega_0 t-n\pi}\,. \tag{2.2}$$

It is easy to show that the linear space $BL^2_{\omega_0}$ is an algebra $U^*_{\omega_0}$ and an algebra $U^{\circledast}_{\omega_0}$ with the products defined as (1) and (2) respectively. If the functions $f(t)$ and $g(t)$ belong to $BL^2_{\omega_0}$, then

$$\begin{aligned}(f*g)&\in BL^2_{\omega_0}\\ (f\circledast g)&\in BL^2_{\omega_0}\,.\end{aligned} \tag{2.3}$$

According to the sampling theorem, the function $f(t)$ has the following discrete representation

$$f(t)=\sum_{-\infty}^{\infty} f_n a_n(t) \tag{2.4}$$

and

$$f_n=f\left(\frac{n\pi}{\omega_0}\right)$$

and

$$a_n(t)=\frac{\sin(\omega_0 t-n\pi)}{(\omega_0 t-n\pi)}$$

We can therefore write, from definition (4)

$$\begin{aligned}(f*g)(\tau)&=\int f(t)\,g(\tau-t)\,dt\\ &=\int\sum_{n,m} f_n g_m a_n(t)\,a_m(\tau-t)\,dt\,.\end{aligned} \tag{2.5}$$

Since $(f*g)$ is a function belonging to BL^2, it is possible to sample it at points π/ω_0 apart. We then obtain the representative vector

$$(f*g)_{\frac{k\pi}{\omega_0}}=\int\sum_{n,m} f_n g_m a_n(t)\,a_{k-m}(t)\,dt \tag{2.6}$$

where use has been made of the identity

$$\mathrm{a}_i\left(\frac{k\pi}{\omega_0}\pm\mathrm{t}\right)=\mathrm{a}_{\pm(i-k)}(\mathrm{t})\,. \tag{2.7}$$

The set $\{a_n\}$ is an orthogonal set with

$$\int a_n(t)\,a_m(t)\,dt=\frac{\pi}{\omega_0}\,\delta_{n,m}\,. \tag{2.8}$$

Therefore, from (2.8) and (2.6), interchanging the order of integration, we obtain

$$(f*g)_k=\frac{\pi}{\omega_0}\sum_{j_n} f_n g_{k-n}=\frac{\pi}{\omega_0}\left(\ldots,\sum f_n g_{-n},\sum f_k g_{1-n},\ldots\right). \tag{2.9}$$

In a completely analogous way, starting from definition (4), we get

$$(f \circledast g)_k = \frac{\pi}{\omega_0} \sum_n f_n g_{n+k} = \frac{\pi}{\omega_0} (\ldots, \sum f_n g_n, \sum f_n g_{n+1}, \ldots).$$

It is therefore clear that the linear algebras B^* and $B^{\circledast}$ defined by Eqs. (13), (14) are a representation of $U^*_{\omega_0}$ and $U^{\circledast}_{\omega_0}$ in the Whittaker basis. This point of view offers a direct physical meaning to the discrete algebras B^* and $B^{\circledast}$ in terms of band-limited signals. The restriction to finite dimensional algebras has again a physical meaning (though definitions (13), (14) must, in that case, be conveniently changed).

In fact one can consider only those signals in $BL^2_{\omega_0}$ whose energy is "principally" contained in a finite time interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$. It is then possible to make the Shannon assumption that there are only a limited number S (Shannon number) of "physically meaningful" sampling points

$$S = \frac{T\omega_0}{\pi}. \tag{2.11}$$

In this way we are able to describe the band-limited signals, further limited to $\left(-\frac{T}{2}, \frac{T}{2}\right)$, as the finite vectors

$$f = (f_{-s}, f_{-s+1}, \ldots, f_0, f_1, \ldots, f_s). \tag{2.12}$$

However, we must observe that (2.12) is only an approximation for the associated function: it is mathematically impossible to reconcile the idea of a function being limited in both the frequency and time domains because of the uncertainty principle (Landau, Pollack, 1961). For the same reason band-limited functions do not have a discrete image in the isomorphic Fourier space.

Estimates of the errors induced by the Shannon assumption can be found in the literature (e.g. Thomas, 1963). They tell how closely the finite algebras B^* and $B^{\circledast}$ represent band-limited signals that are time limited to $\left(-\frac{T}{2}, \frac{T}{2}\right)$. In this paper, however, we rather assume infinite dimensional spaces, remembering that, whenever algebraically convenient, the reduction to finite dimensions still holds a clear physical meaning.

Appendix III

The check of associativity and commutativity is trivial using either definitions (14), (13) or the Fourier equivalent relationships for the corresponding band-limited functions. We shall rather consider the commutative properties of B^* and $B^{\circledast}$, introducing another algorithm. From the vectors f and g (real-valued).

$$\begin{aligned} f &= (\ldots, f_{-1}, f_0, f_1, \ldots) \\ g &= (\ldots, g_{-1}, g_0, g_1, \ldots) \end{aligned} \tag{3.1}$$

the following matrix is formed

$$\begin{pmatrix} & & \cdots & & \\ \cdots & f_{-1}g_{-1} & f_{-1}g_0 & f_{-1}g_1 & \cdots \\ \cdots & f_0\ g_{-1} & f_0\ g_0 & f_0\ g_1 & \cdots \\ & \cdots & f_1\ g_0 & f_1\ g_1 & \cdots \\ & & \cdots & & \end{pmatrix} = (fg)_{nm} = f_n g_m. \tag{3.2}$$

It is easy to see that, according to definition (14),

$$(f \circledast g)_k = \mathrm{tr}_k(fg), \tag{3.3}$$

where tr_k indicates the sum of the matrix elements which are located on a diagonal parallel to the main one $((fg)_{i,i})$ but displaced to the right by k. Of course $\mathrm{tr}_{k=0} \equiv \mathrm{tr}$. On the other hand

$$(f * g)_k = \mathrm{tr}^*_k(fg), \tag{3.4}$$

where tr^*_k indicates the operation of adding all the matrix elements parallel to the main antidiagonal $(fg)_{i,-i}$ and displaced to the right by k.

It is clear that

$$(gf) = (fg)^t, \tag{3.5}$$

where t indicates the transposed matrix. Since the $(\mathrm{tr}^*)_k$ operation is invariant under transposition when $(\mathrm{tr})_k$ is not $(\mathrm{tr}_k(gf) = \mathrm{tr}_{-k}(gf)^t)$, we conclude that convolution is commutative and correlation is not. Similar arguments can be used with respect to associativity. Interestingly enough, the formal operations defined through Eqs. (3.2, (3.3), (3.4) represent also an easy algorithm for computing convolution and correlation. Simple nets (in the sense of Willshaw *et al.*, 1969) able to mimic in this way the holographic memory scheme [Eq. (60)] are readily suggested.

It is also possible to obtain the number of distinct classes on not-associative products (as in the case of $B^{\circledast}$) of n elements. One obtains the recurrence relation

$$\begin{aligned} F(n) &= F(1)\,F(n-1) + F(2)\,F(n-2) + \cdots + F(n-1)\,F(1) \\ &= \sum_{k=1}^{n-1} F(k)\,F(n-k) \end{aligned} \tag{3.6}$$

which gives

$$F(n) = \frac{1}{n}\binom{2n-2}{n-1}. \tag{3.7}$$

We have $F(1) = F(2) = 1$; for $F(3) = 2$ there are the two products $(ab)\,c$ and $a(bc)$. Of course, the number of distinct products decreases, in the case of $B^{\circledast}$, if even vectors are present.

Appendix IV

It is only necessary to check that the set of all linear combinations of "noise-like unities", as defined by (44), is closed.

In fact, from linear combinations of n^α we can obtain only noise-like vectors, that is vectors ψ such that

$$(\psi \circledast \psi)_i = \delta_{i,0}.$$

Linear combinations as well as correlations or convolutions between noise-like vectors give again noise-like vectors. The set of all linear combinations of "noise-like unities" is then – when completed with $\delta_{i,0}$ – a subalgebra of $B^{\circledast}$.

Appendix V

Centralizer and center of B^* and $B^{\circledast}$

From Schur lemma, the centralizer C' of an algebra is an associative division ring. C' is defined as the set of all linear transformations T such that

$$\begin{aligned} R_y T &= T R_y \\ L_x T &= T L_x. \end{aligned} \tag{5.1}$$

Therefore, if S, T belong to C' they commute;

$$ST = TS. \tag{5.2}$$

We now state the following results:

The endomorphisms V which belong to C' of B^* are those V such that, in matrix notation, (5.3)

$$V_{l-i,s-i} = V_{l,s}.$$

The endomorphisms W which belong to C' of $B^{\circledast}$ are those W such that, in matrix notation, (5.4)

$$W_{i-l,i-s} = W_{l,s}$$
$$W_{l+i,s+i} = W_{l,s}.$$

The matrix representation of the centralizer of $B^{\circledast}$ shows, therefore, diagonals equal in pairs with respect to the principal one. For example

$$\begin{pmatrix} & & & \cdots & & & & \\ \cdots & c & a & b & a & c & \cdots & \\ & \cdots & c & a & b & a & c & \cdots \\ & & \cdots & c & a & b & a & c \; \cdots \\ & & & \cdots & & & & \end{pmatrix}. \tag{5.5}$$

The center C of an algebra A is defined as the set of all quantities $c \in A$ such that the commutative and associative laws hold whenever c is one of the factors. Of course $B^* \equiv C$. With respect to $B^{\circledast}$ the definition implies

$$\begin{aligned} &R_c^{\circledast} = L_c^{\circledast} \\ &R_c^{\circledast} R_g^{\circledast} = R_g^{\circledast} R_c^{\circledast} = R_{cg}^{\circledast}, \quad \forall g \in B^{\circledast}, \quad \forall c \in C. \end{aligned} \tag{5.6}$$

The center of $B^{\circledast}$ is thereby found to consist of all "alternant" vectors of the form

$$a = (\ldots a, b, a, b, a, b, \ldots). \tag{5.7}$$

"Alternant" vectors are an ideal of $B^{\circledast}$ and they are *the only ones which commute and associate with all other vectors.* A special subset is the set of all "constant" vectors

$$k = (\ldots b, b, b, b \ldots). \tag{5.8}$$

Interestingly enough, the left unity e of $B^{\circledast}$ does not belong to C. We observe further that

$$[c, x]^{\circledast} \equiv 0 \quad \text{for} \quad x \in B^{\circledast}, c \in C. \tag{5.9}$$

Appendix VI

The correlation algebra is characterized by the relationships

$$R_x^{\circledast} R_y^{\circledast} = L_{yx}^{\circledast} \leftrightarrow (a \circledast x) \circledast y = (y \circledast x) \circledast a, \tag{6.1}$$

$$R_y^{\circledast} L_x^{\circledast} = R_{xy}^{\circledast} \leftrightarrow x \circledast (a \circledast y) = a \circledast (x \circledast y). \tag{6.2}$$

The last is implied, through a theorem by Albert, by the existence of a left unity in the correlation algebra. Relation (6.1) gives

$$(R_x^{\circledast})^2 = L_{x^2}^{\circledast}. \tag{6.3}$$

(6.2) gives

$$R_x^{\circledast} L_x^{\circledast} = R_{x^2}^{\circledast} \tag{6.4}$$

Relationships (6.3), (6.4) can be compared, for instance, with the identities

$$(R_x)^2 = R_{x^2}, \tag{6.5}$$

$$(L_x)^2 = L_{x^2}, \tag{6.6}$$

which, according to Table 1, characterize the alternative algebras.

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